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MATH 253 Honors EXAM 1 Fall 2002

Sections 201-202 Solutions P. Yasskin

Multiple Choice: (8 points each)

1. Find the area of the triangle with vertices $A = (2,-1,3)$, $B = (3,-2,1)$, and $C = (1,3,2)$.

a. $\dfrac{\sqrt{99}}{2}$ correctchoice

b. $\dfrac{75}{2}$

c. $\sqrt{99}$

d. 75

e. $\dfrac{\sqrt{75}}{2}$

$\overrightarrow{AB} = B - A = (1,-1,-2) \qquad \overrightarrow{AC} = C - A = (-1,4,-1)$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -1 & -2 \\ -1 & 4 & -1 \end{vmatrix} = \hat{\imath}(9) - \hat{\jmath}(-3) + \hat{k}(3) = (9,3,3)$$

$$Area_\Delta = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \frac{1}{2}\sqrt{81+9+9} = \frac{\sqrt{99}}{2}$$

2. A light ray travelling along the line $x = 2 - t$, $y = 3 + 2t$, $z = 4 - 2t$ strikes a mirror in the plane $x + y - z = 7$ at the point $(0,7,0)$. Find the angle of incidence, i.e. the angle between the tangent vector to the line and the normal to the plane.

a. $\arccos\left(\dfrac{1}{\sqrt{2}}\right)$

b. $\arccos\left(\dfrac{1}{2}\right)$

c. $\arccos\left(\dfrac{1}{\sqrt{3}}\right)$ correctchoice

d. $\arccos\left(\dfrac{\sqrt{3}}{2}\right)$

e. $\arccos\left(\dfrac{1}{3}\right)$

$\vec{v} = (-1,2,-2) \qquad \vec{N} = (1,1,-1) \qquad |\vec{v}| = 3 \qquad \left|\vec{N}\right| = \sqrt{3} \qquad \vec{v}\cdot\vec{N} = -1+2+2 = 3$

$$\cos\theta = \frac{\vec{v}\cdot\vec{N}}{|\vec{v}|\left|\vec{N}\right|} = \frac{3}{3\sqrt{3}} = \frac{1}{\sqrt{3}}$$

3. Find the arc length of the curve $\vec{r}(t) = \left(\frac{2}{3}t^3, t^2, t\right)$ for $0 \le t \le 1$.

HINT: Look for a perfect square.

a. $\frac{5}{6}$

b. 1

c. $\frac{5}{4}$

d. $\frac{5}{3}$ correctchoice

e. $\frac{5}{2}$

$\vec{v} = (2t^2, 2t, 1) \qquad |\vec{v}| = \sqrt{4t^4 + 4t^2 + 1} = 2t^2 + 1$

$L = \int_0^1 |\vec{v}|\,dt = \int_0^1 (2t^2 + 1)\,dt = \left[\frac{2}{3}t^3 + t\right]_0^1 = \frac{5}{3}$

4. If a jet plane is travelling from East to West directly above the Equator, in which direction does the binormal $\hat{B}$ point?

a. North

b. South correctchoice

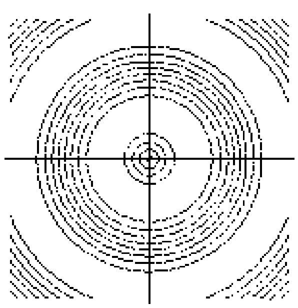
c. East

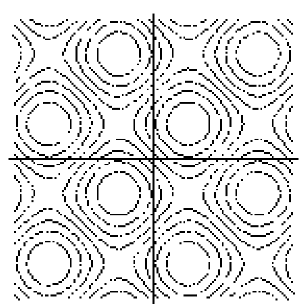
d. West

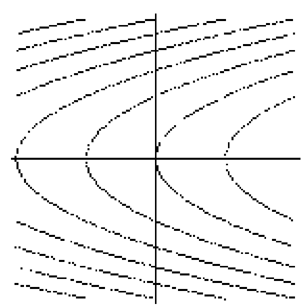
e. Down

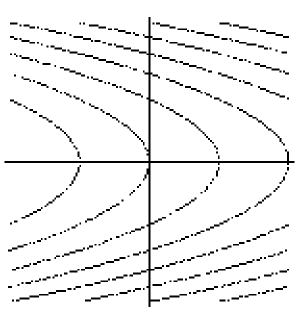
$\hat{T}$ points West since it is travelling West, $\hat{N}$ points Down since its circular path bends toward the center of the earth. So $\hat{B} = \hat{T} \times \hat{N}$ points South by the right hand rule.

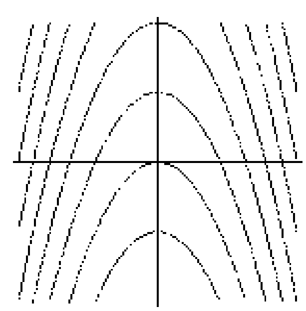
5. Which of the following is the contour plot of $z = x + y^2$?

a 

b 

c 

d  correctchoice

e 

The level curve $z = 0$ is $x = -y^2$ which only appears in figure (d).

6. Compute the gradient of the function $f(x,y) = x^3 \cos 2y$ at the point $(x,y) = \left(2, \frac{\pi}{6}\right)$.

a. $\left(6, -8\sqrt{3}\right)$ correctchoice
b. $\left(6\sqrt{3}, -8\right)$
c. $\left(-6\sqrt{3}, 4\right)$
d. $\left(-6, 8\sqrt{3}\right)$
e. $\left(6, -4\sqrt{3}\right)$

$\vec{\nabla} f = (3x^2 \cos 2y, -2x^3 \sin 2y) \qquad \vec{\nabla} f\Big|_{\left(2,\frac{\pi}{6}\right)} = \left(12 \cos \frac{\pi}{3}, -16 \sin \frac{\pi}{3}\right) = \left(6, -8\sqrt{3}\right)$

7. Find the equation of the plane tangent to the graph of $z = xy^2$ at the point $(x,y) = (2,1)$.

a. $z = x + 4y - 6$
b. $z = x + 4y - 4$ correctchoice
c. $z = -x - 4y + 2$
d. $z = -x - 4y + 6$
e. $z = -x - 4y + 8$

$f(x,y) = xy^2 \qquad f_x(x,y) = y^2 \qquad f_y(x,y) = 2xy$
$f(2,1) = 2 \qquad f_x(2,1) = 1 \qquad f_y(2,1) = 4$
$z = f(2,1) + f_x(2,1)(x-2) + f_y(2,1)(y-1) = 2 + (x-2) + 4(y-1) = x + 4y - 4$

8. If $g(3,4) = 7$ and $\frac{\partial g}{\partial x}(3,4) = 0.5$ and $\frac{\partial g}{\partial y}(3,4) = -0.2$, estimate $g(2.8, 4.3)$.

a. $.16$
b. $.54$
c. 7.16
d. 7.54
e. 6.84 correctchoice

$$g(x,y) \approx g(a,b) + \frac{\partial g}{\partial x}(a,b)(x-a) + \frac{\partial g}{\partial y}(a,b)(y-b)$$
$$g(2.8,4.3) \approx g(3,4) + \frac{\partial g}{\partial x}(3,4)(2.8-3) + \frac{\partial g}{\partial y}(3,4)(4.3-4)$$
$$= 7 + 0.5(-.2) - 0.2(.3) = 6.84$$

9. (18 points) Suppose $f = \frac{xz}{y}$ where $x = x(u,v)$, $y = y(u,v)$ and $z = z(u,v)$.

Further suppose $x(1,2) = 4$, $y(1,2) = 6$, $z(1,2) = 3$ and

$$\frac{\partial x}{\partial u}(1,2) = 3,\ \frac{\partial y}{\partial u}(1,2) = -2,\ \frac{\partial z}{\partial u}(1,2) = 5,\ \frac{\partial x}{\partial v}(1,2) = 4,\ \frac{\partial y}{\partial v}(1,2) = 3,\ \frac{\partial z}{\partial v}(1,2) = -3$$

Compute $\frac{\partial f}{\partial v}(1,2)$.

By the chain rule:

$$\left.\frac{\partial f}{\partial v}\right|_{(1,2)} = \left.\frac{\partial f}{\partial x}\right|_{(4,6,3)} \left.\frac{\partial x}{\partial v}\right|_{(1,2)} + \left.\frac{\partial f}{\partial y}\right|_{(4,6,3)} \left.\frac{\partial y}{\partial v}\right|_{(1,2)} + \left.\frac{\partial f}{\partial z}\right|_{(4,6,3)} \left.\frac{\partial z}{\partial v}\right|_{(1,2)}$$

But:

$$\frac{\partial f}{\partial x} = \frac{z}{y} \qquad \frac{\partial f}{\partial y} = -\frac{xz}{y^2} \qquad \frac{\partial f}{\partial z} = \frac{x}{y}$$

$$\left.\frac{\partial f}{\partial x}\right|_{(4,6,3)} = \frac{3}{6} = \frac{1}{2} \qquad \left.\frac{\partial f}{\partial y}\right|_{(4,6,3)} = -\frac{12}{36} = -\frac{1}{3} \qquad \left.\frac{\partial f}{\partial z}\right|_{(4,6,3)} = \frac{4}{6} = \frac{2}{3}$$

So:

$$\left.\frac{\partial f}{\partial v}\right|_{(1,2)} = \frac{1}{2} \cdot 4 - \frac{1}{3} \cdot 3 + \frac{2}{3} \cdot (-3) = 2 - 1 - 2 = -1$$

10. (18 points) Find the point in the first octant on the graph of $xy^2z^3 = 2$ which is closest to the origin.

Since $x = \frac{2}{y^2z^3}$ and we minimize $f = x^2 + y^2 + z^2 = \frac{4}{y^4z^6} + y^2 + z^2$.

$$f_y = \frac{-16}{y^5z^6} + 2y = 0 \quad \Rightarrow \quad y^6z^6 = 8 \quad \Rightarrow \quad yz = 2^{1/2}$$

$$f_z = \frac{-24}{y^4z^7} + 2z = 0 \quad \Rightarrow \quad y^4z^8 = 12 \quad \Rightarrow \quad yz^2 = 12^{1/4}$$

$$\Rightarrow \quad z = \frac{yz^2}{yz} = \frac{12^{1/4}}{2^{1/2}} = 3^{1/4} \quad \Rightarrow \quad y = \frac{2^{1/2}}{z} = \frac{2^{1/2}}{3^{1/4}}$$

$$\Rightarrow \quad x = \frac{2}{y^2z^3} = \frac{2}{\left(\frac{2^{1/2}}{3^{1/4}}\right)^2 (3^{1/4})^3} = \frac{1}{3^{1/4}}$$

So the point is $\left(\frac{1}{3^{1/4}}, \frac{2^{1/2}}{3^{1/4}}, 3^{1/4}\right)$