1-4	/40
5	/15
6	/15
7	/15
8	/15

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MATH 253 Honors EXAM 2 Fall 2002

Sections 201-202 Solutions P. Yasskin

Multiple Choice: (10 points each) Work Out: (15 points each)

1. Compute $\int_0^2 \int_x^{2x} xy\,dy\,dx$.

a. 2
b. 4
c. 6 correctchoice
d. 8
e. 10

$$\int_0^2 \int_x^{2x} xy\,dy\,dx = \int_0^2 \left[\frac{xy^2}{2} \right]_{y=x}^{2x} dx = \int_0^2 \left(\frac{x \cdot 4x^2}{2} - \frac{x \cdot x^2}{2} \right) dx = \int_0^2 \frac{3x^3}{2}\,dx = \left[\frac{3x^4}{8} \right]_{x=0}^{2} = 6$$

2. Compute $\iint \sin(x^2+y^2)\,dx\,dy$ over the region in the first quadrant between the circles $x^2+y^2=\pi$ and $x^2+y^2=2\pi$.

a. $-\frac{\pi}{2}$ correctchoice
b. $\frac{\pi}{2}$
c. $-\frac{\pi}{4}$
d. $\frac{\pi}{4}$
e. $\frac{\pi^2}{4}$

$$\iint \sin(x^2+y^2)\,dx\,dy = \int_0^{\pi/2} \int_{\sqrt{\pi}}^{\sqrt{2\pi}} \sin(r^2)\,r\,dr\,d\theta = \frac{\pi}{2} \int_{\sqrt{\pi}}^{\sqrt{2\pi}} \sin(r^2)\,r\,dr \qquad \text{Let } u = r^2 \quad du = 2r\,dr$$

$$= \frac{\pi}{4} \int_{\pi}^{2\pi} \sin(u)\,du = -\frac{\pi}{4} \cos u \Big|_{\pi}^{2\pi} = -\frac{\pi}{4}(1 - -1) = -\frac{\pi}{2}$$

3. Find the mass of the upper half ($z \ge 0$) of the cylinder $x^2 + z^2 \le 4$ for $0 \le y \le 3$ if the density is $\delta = 6z$.

- **a.** 24π
- **b.** 36π
- **c.** 24
- **d.** 36
- **e.** 96 correctchoice

$$M = \iiint \delta\, dV = \int_{-2}^{2}\int_{0}^{3}\int_{0}^{\sqrt{4-x^2}} 6z\, dz\, dy\, dx = \int_{-2}^{2}\int_{0}^{3}\left[3z^2\right]_{z=0}^{\sqrt{4-x^2}} dy\, dx = \int_{-2}^{2}\int_{0}^{3} 3(4-x^2)\, dy\, dx$$

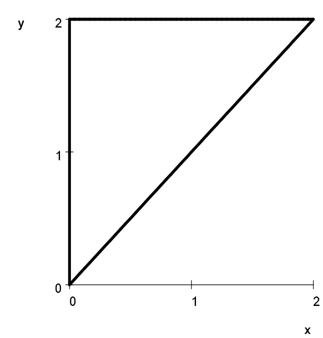
$$= 9\int_{-2}^{2}(4-x^2)\, dx = 9\left[4x - \frac{x^3}{3}\right]_{-2}^{2} = 18\left[8 - \frac{8}{3}\right] = 18 \cdot 8 \cdot \frac{2}{3} = 6 \cdot 16 = 96$$

4. Compute $\int_{0}^{2}\int_{x}^{2} \frac{x}{1+y^3}\, dy\, dx$.

- **a.** $\frac{1}{2}\ln 9$
- **b.** $\frac{1}{3}\ln 9$
- **c.** $\frac{1}{6}\ln 9$ correctchoice
- **d.** $\frac{1}{6}\ln 3 - \frac{1}{6}\pi\sqrt{3} + \frac{2}{9}$
- **e.** $\frac{1}{6}\ln 3 + \frac{1}{6}\pi\sqrt{3} + \frac{2}{9}$

$$\int_{0}^{2}\int_{x}^{2} \frac{x}{1+y^3}\, dy\, dx = \int_{0}^{2}\int_{0}^{y} \frac{x}{1+y^3}\, dx\, dy = \frac{1}{2}\int_{0}^{2} \frac{x^2}{1+y^3}\Big|_{x=0}^{y} dy$$

$$= \frac{1}{2}\int_{0}^{2} \frac{y^2}{1+y^3}\, dy = \frac{1}{6}\ln|1+y^3|\Big|_{y=0}^{2} = \frac{1}{6}\ln 9$$



5. (15 points) Find the point in the first octant on the graph of $xy^2z^3 = 2$ which is closest to the origin. You must use Lagrange multipliers.

Minimize $f = x^2 + y^2 + z^2$ subject to $g = xy^2z^3 = 2$.

$\vec{\nabla} f = (2x, 2y, 2z) \qquad \vec{\nabla} g = (y^2z^3, 2xyz^3, 3xy^2z^2)$

$\vec{\nabla} f = \lambda \vec{\nabla} g \quad \Rightarrow \quad 2x = \lambda y^2 z^3, \quad 2y = \lambda 2xyz^3, \quad 2z = \lambda 3xy^2z^2$

$\lambda = \frac{2x}{y^2z^3} = \frac{1}{xz^3} = \frac{2}{3xy^2z} \quad \Rightarrow \quad 2x^2 = y^2, \quad 3x^2 = z^2$

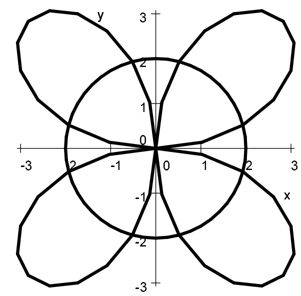
$\Rightarrow \quad y = \sqrt{2}\,x, \quad z = \sqrt{3}\,x$

$2 = xy^2z^3 = x\left(\sqrt{2}\,x\right)^2\left(\sqrt{3}\,x\right)^3 = 6\sqrt{3}\,x^6 \quad \Rightarrow \quad x^6 = \frac{2}{6\sqrt{3}} = 3^{-3/2}$

$\Rightarrow \quad x = 3^{-1/4} \quad \Rightarrow \quad y = \sqrt{2}\,3^{-1/4}, \quad z = \sqrt{3}\,3^{-1/4} = 3^{1/4}$

So the point is $\left(\frac{1}{3^{1/4}}, \frac{2^{1/2}}{3^{1/4}}, 3^{1/4}\right)$

6. (15 points) Find the area inside one petal of the 4-petal rose $r = 4\sin(2\theta)$ but outside the circle $r = 2$.



$$4\sin(2\theta) = 2 \quad\Rightarrow\quad \sin(2\theta) = \frac{1}{2} \quad\Rightarrow\quad 2\theta = \frac{\pi}{6}, \frac{5\pi}{6} \quad\Rightarrow\quad \theta = \frac{\pi}{12}, \frac{5\pi}{12}$$

$$A = \int_{\pi/12}^{5\pi/12}\int_{2}^{4\sin(2\theta)} r\,dr\,d\theta = \frac{1}{2}\int_{\pi/12}^{5\pi/12}\left[r^2\right]_2^{4\sin(2\theta)} d\theta = \frac{1}{2}\int_{\pi/12}^{5\pi/12}(16\sin^2(2\theta) - 4)\,d\theta$$

$$= \frac{1}{2}\int_{\pi/12}^{5\pi/12}(8(1-\cos(4\theta)) - 4)\,d\theta = \frac{1}{2}\int_{\pi/12}^{5\pi/12}(4 - 8\cos(4\theta))\,d\theta = \frac{1}{2}\left[4\theta - 2\sin(4\theta)\right]_{\theta=\pi/12}^{5\pi/12}$$

$$= \left(2\frac{5\pi}{12} - \sin\left(4\frac{5\pi}{12}\right)\right) - \left(2\frac{\pi}{12} - \sin\left(4\frac{\pi}{12}\right)\right) = \left(\frac{5\pi}{6} - \sin\left(\frac{5\pi}{3}\right)\right) - \left(\frac{\pi}{6} - \sin\left(\frac{\pi}{3}\right)\right)$$

$$= \frac{4\pi}{6} - \sin\left(\frac{5\pi}{3}\right) + \sin\left(\frac{\pi}{3}\right) = \frac{2\pi}{3} - \left(-\frac{\sqrt{3}}{2}\right) + \frac{\sqrt{3}}{2} = \frac{2\pi}{3} + \sqrt{3}$$

7. (15 points) Consider the upper half ($z \geq 0$) of the sphere $x^2 + y^2 + z^2 \leq 4$. Find the mass and center of mass of the hemisphere if the density is $\delta = 5z$. Use symmetry where appropriate.

$$M = \iiint \delta\,dV = \iiint 5z\,dx\,dy\,dz = \int_0^{2\pi}\int_0^{\pi/2}\int_0^2 5\rho\cos\varphi\,\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta = 10\pi\int_0^{\pi/2}\int_0^2 \rho^3\cos\varphi\sin\varphi\,d\rho\,d\varphi$$

$$= 10\pi\left[\frac{\rho^4}{4}\right]_{\rho=0}^{2}\left[\frac{\sin^2\varphi}{2}\right]_{\varphi=0}^{\pi/2} = 10\pi[4]\left[\frac{1}{2}\right] = 20\pi$$

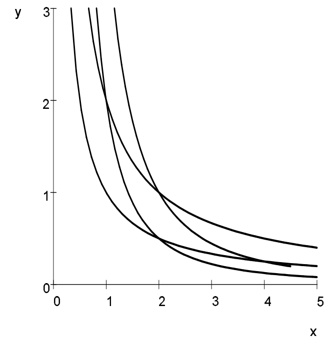
$$M_{xy} = \iiint z\delta\,dV = \iiint 5z^2\,dx\,dy\,dz = \int_0^{2\pi}\int_0^{\pi/2}\int_0^2 5\rho^2\cos^2\varphi\,\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta = 10\pi\int_0^{\pi/2}\int_0^2 \rho^4\cos^2\varphi\sin\varphi\,d\rho$$

$$= 10\pi\left[\frac{\rho^5}{5}\right]_{\rho=0}^{2}\left[\frac{-\cos^3\varphi}{3}\right]_{\varphi=0}^{\pi/2} = 10\pi\left[\frac{32}{5}\right]\left[\frac{1}{3}\right] = \frac{64\pi}{3}$$

$$\bar{z} = \frac{M_{xy}}{M} = \frac{64\pi}{3 \cdot 20\pi} = \frac{16}{15}$$ By symmetry $\bar{x} = 0$ and $\bar{y} = 0$.

8. (15 points) Compute $\iint_R x^2 y \; dx\, dy$ over the diamond shaped region R bounded by

$$y = \frac{1}{x}, \qquad y = \frac{2}{x}, \qquad y = \frac{2}{x^2}, \qquad y = \frac{4}{x^2}$$



FULL CREDIT for integrating in the curvilinear coordinates
$u = xy$ and $v = x^2 y$. (Solve for x and y.)

HALF CREDIT for integrating in rectangular coordinates.

METHOD 1: Integrand: $x^2 y = v$ Limits: $1 \le u \le 2 \quad 2 \le v \le 4$

Solve for x and y : $\frac{v}{u} = \frac{x^2 y}{xy} = x \qquad y = \frac{u}{x} = u\frac{u}{v} = \frac{u^2}{v}$

Summary: $x = u^{-1}v, \; y = u^2 v^{-1}$

$$J = \left\| \begin{matrix} -u^{-2}v & u^{-1} \\ 2uv^{-1} & -u^2 v^{-2} \end{matrix} \right\| = |u^{-2}vu^2v^{-2} - u^{-1}2uv^{-1}| = \left|\frac{1}{v} - \frac{2}{v}\right| = \left|-\frac{1}{v}\right| = \frac{1}{v}$$

$$\iint_R x^2 y \; dx\; dy = \int_2^4 \int_1^2 v\,\frac{1}{v}\; du\; dv = \int_2^4 \int_1^2 1\; du\; dv = (2-1)(4-2) = 2$$

METHOD 2: Find intersections:

$$\frac{2}{x} = \frac{2}{x^2} \;\Rightarrow\; 2x^2 = 2x \;\Rightarrow\; x = 1, \qquad \frac{1}{x} = \frac{2}{x^2} \;\Rightarrow\; x^2 = 2x \;\Rightarrow\; x = 2$$

$$\frac{2}{x} = \frac{4}{x^2} \;\Rightarrow\; 2x^2 = 4x \;\Rightarrow\; x = 2, \qquad \frac{1}{x} = \frac{4}{x^2} \;\Rightarrow\; x^2 = 4x \;\Rightarrow\; x = 4$$

So the integral breaks into two pieces:

$$\iint_R x^2 y \; dx\,dy = \int_1^2 \int_{2/x^2}^{2/x} x^2 y \; dy \; dx + \int_2^4 \int_{1/x}^{4/x^2} x^2 y \; dy \; dx = \int_1^2 \left[x^2 \frac{y^2}{2}\right]_{y=2/x^2}^{2/x} dx + \int_2^4 \left[x^2 \frac{y^2}{2}\right]_{y=1/x}^{4/x^2} dx$$

$$= \frac{1}{2}\int_1^2 \left[x^2 \frac{4}{x^2}\right] - \left[x^2 \frac{4}{x^4}\right] dx + \frac{1}{2}\int_2^4 \left[x^2 \frac{16}{x^4}\right] - \left[x^2 \frac{1}{x^2}\right] dx$$

$$= \frac{1}{2}\int_1^2 \left(4 - \frac{4}{x^2}\right) dx + \frac{1}{2}\int_2^4 \left(\frac{16}{x^2} - 1\right) dx = \frac{1}{2}\left[4x + \frac{4}{x}\right]_1^2 + \frac{1}{2}\left[-\frac{16}{x} - x\right]_2^4$$

$$= \frac{1}{2}([8+2] - [4+4] + [-4-4] - [-8-2]) = \frac{1}{2}(10 - 8 - 8 + 10) = 2$$