Name____________________ ID___________ Section_____

MATH 253 Honors — Final Exam — Fall 2002

Sections 201-202 — Solutions — P. Yasskin

1-5	/40	8	/20
6	/10	9	/10
7	/10	10	/10

Multiple Choice: (8 points each)   Work Out: (points indicated)

1. Find the volume of the parallelepiped with edges $\vec{u} = (1,0,3)$, $\vec{v} = (0,2,-1)$ and $\vec{w} = (2,0,2)$.

a. -8
b. -4
c. 4
d. 8 correctchoice
e. 16

$$V = |\vec{u} \cdot \vec{v} \times \vec{w}| = \left\| \begin{matrix} 1 & 0 & 3 \\ 0 & 2 & -1 \\ 2 & 0 & 2 \end{matrix} \right\| = |-8| = 8$$

2. Duke Skywater is flying the Millenium Eagle through a polaron field. His galactic coordinates are $(2300, 4200, 1600)$ measured in lightseconds and his velocity is $\vec{v} = (.2, .3, .4)$ measured in lightseconds per second. He measures the strength of the polaron field is $p = 274$ milliwookies and its gradient is $\vec{\nabla} p = (3,2,2)$ milliwookies per lightsecond. Assuming a linear approximation for the polaron field and that his velocity is constant, how many seconds will Duke need to wait until the polaron field has grown to 286 milliwookies?

a. 2
b. 3
c. 4
d. 6 correctchoice
e. 12

The derivative along Duke's path is

$$\frac{dp}{dt} = \vec{v} \cdot \vec{\nabla} p = (.2, .3, .4) \frac{\text{lightseconds}}{\text{second}} \cdot (3,2,2) \frac{\text{milliwookies}}{\text{lightsecond}} = .6 + .6 + .8 = 2 \frac{\text{milliwookies}}{\text{second}}$$

So the polaron field increases 2 milliwookies each second.
To increase 12 milliwookies, it will take 6 seconds.

3. Find the plane tangent to the hyperbolic paraboloid $x = yz$ at the point $P = (6,3,2)$. Which of the following points does **not** lie on this plane?

a. $(-6,0,0)$
b. $(0,3,0)$
c. $(0,0,2)$
d. $(-1,1,1)$
e. $(1,-1,-1)$ correctchoice

Let $f(y,z) = yz$. Then $f_y = z$ and $f_z = y$.

At the point $(y,z) = (3,2)$, we have $f(3,2) = 6, f_y(3,2) = 2$ and $f_z(3,2) = 3$.

So the plane tangent to $x = f(y,z)$ at $(y,z) = (3,2)$ is

$x = f_{\tan}(y,z) = f(3,2) + f_y(3,2)(y-3) + f_z(3,2)(z-2) = 6 + 2(y-3) + 3(z-2)$ or $x = 2y + 3z - 6$

Plugging in each point, we find $(1,-1,-1)$ is not a solution.

4. A airplane is circling with constant speed above Kyle Field along the curve $\vec{r}(t) = (\cos(8\pi t), \sin(8\pi t), 2)$ where distances are in miles and time is in hours. Find the tangential acceleration a_T, where the acceleration is $\vec{a} = a_T\hat{T} + a_N\hat{N}$.

a. 0 correctchoice
b. 8π
c. -8π
d. $64\pi^2$
e. $-64\pi^2$

$\vec{v} = (-8\pi\sin(8\pi t), 8\pi\cos(8\pi t), 0) \qquad \vec{a} = (-64\pi^2\cos(8\pi t), -64\pi^2\sin(8\pi t), 0)$

$|\vec{v}| = \sqrt{64\pi^2} = 8\pi \qquad \hat{T} = (-\sin(8\pi t), \cos(8\pi t), 0) \qquad a_T = \dfrac{d|\vec{v}|}{dt} = 0$

OR $a_T = \vec{a} \cdot \hat{T} = 64\pi^2\cos(8\pi t)\sin(8\pi t) - 64\pi^2\cos(8\pi t)\sin(8\pi t) = 0$

5. Find the volume below the plane $z = 6 - 2y$ above the triangle with vertices $(0,0,0)$, $(1,0,0)$ and $(0,3,0)$.

a. 3
b. 6 correctchoice
c. 9
d. 12
e. 15

$$V = \int_0^1\int_0^{3-3x}(6-2y)\,dy\,dx = \int_0^1[6y-y^2]_0^{3-3x}\,dx = \int_0^1\left[6(3-3x)-(3-3x)^2\right]dx = \int_0^1(9-9x^2)\,dx$$

$$= [9x-3x^3]_0^1 = 9-3 = 6$$

6. (10 points) Find the location and value of the minimum of the function $f(x,y,z) = x^2 + 2y^2 + 3z^2$ on the plane $x + y + z = 11$.

METHOD 1: Lagrange Multipliers:

$f = x^2 + 2y^2 + 3z^2 \qquad \vec{\nabla} f = (2x, 4y, 6z) \qquad g = x + y + z \qquad \vec{\nabla} g = (1,1,1)$

$\vec{\nabla} f = \lambda \vec{\nabla} g \quad \Rightarrow \quad 2x = \lambda, \quad 4y = \lambda, \quad 6z = \lambda \quad \Rightarrow \quad x = \dfrac{\lambda}{2}, \quad y = \dfrac{\lambda}{4}, \quad z = \dfrac{\lambda}{6}$

$x + y + z = \dfrac{\lambda}{2} + \dfrac{\lambda}{4} + \dfrac{\lambda}{6} = 11 \quad \Rightarrow \quad 6\lambda + 3\lambda + 2\lambda = 11 \cdot 12 \quad \Rightarrow \quad \lambda = 12$

$x = 6, \quad y = 3, \quad z = 2 \qquad f(6,3,2) = 36 + 2 \cdot 9 + 3 \cdot 4 = 66$

METHOD 2: Eliminate a Variable:

$z = 11 - x - y \quad \Rightarrow \quad f = x^2 + 2y^2 + 3(11 - x - y)^2$

$f_x = 2x - 6(11 - x - y) = 8x + 6y - 66 = 0 \qquad f_y = 4y - 6(11 - x - y) = 6x + 10y - 66 = 0$

Cross multiply: $80x + 60y = 660 \qquad 36x + 60y = 396 \quad \Rightarrow \quad 44x = 264 \quad \Rightarrow \quad x = 6$

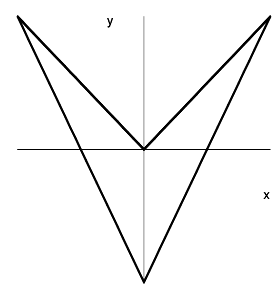
Substitute back into f_y: $36 + 10y - 66 = 0 \quad \Rightarrow \quad y = 3$

Substitute back: $z = 11 - 6 - 3 = 2 \qquad f(6,3,2) = 36 + 2 \cdot 9 + 3 \cdot 4 = 66$

7. (10 points) Consider the region between the curves

$y = 2|x| - 2$ and $y = |x|$.

If the density is $\delta = 2 + 2y$ compute the mass and y-component of the center of mass of this region.
(7 points for setup. 3 points for evaluation.)



Find positive intersection: $2x - 2 = x \quad \Rightarrow \quad x = 2$

Use symmetry to double the integral for positive x.

$$M = 2\int_0^2 \int_{2x-2}^{x} (2 + 2y)\,dy\,dx = 2\int_0^2 [2y + y^2]_{2x-2}^{x}\,dx = 2\int_0^2 [2x + x^2] - \left[2(2x-2) + (2x-2)^2\right] dx$$

$$= 2\int_0^2 (6x - 3x^2)\,dx = 2[3x^2 - x^3]_0^2 = 2(12 - 8) = 8$$

$$y\text{-mom} = 2\int_0^2 \int_{2x-2}^{x} y(2 + 2y)\,dy\,dx = 2\int_0^2 \left[y^2 + \frac{2y^3}{3}\right]_{2x-2}^{x} dx$$

$$= 2\int_0^2 \left[x^2 + \frac{2x^3}{3}\right] - \left[(2x-2)^2 + \frac{2(2x-2)^3}{3}\right] dx$$

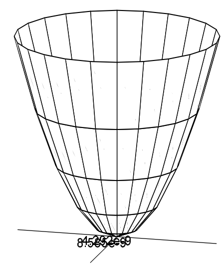
$$= 2\int_0^2 \left(\frac{4}{3} - 8x + 13x^2 - \frac{14}{3}x^3\right) dx = 2\left[\frac{4}{3}x - 4x^2 + 13\frac{x^3}{3} - \frac{14}{3}\frac{x^4}{4}\right]_0^2$$

$$= 2\left(\frac{8}{3} - 16 + \frac{104}{3} - \frac{56}{3}\right) = \frac{16}{3}(1 - 6 + 13 - 7) = \frac{16}{3}$$

$$\bar{y} = \frac{y\text{-mom}}{M} = \frac{16}{3 \cdot 8} = \frac{2}{3}$$

8. $\left(20\text{ points}\right)$ **Stokes' Theorem** states that if S is a nice surface in $\mathbf{R}^3$ and ∂S is its boundary curve traversed counterclockwise as seen from the tip of the normal to S and $\vec{F}$ is a nice vector field on S then

$$\iint_S \vec{\nabla}\times\vec{F}\cdot d\vec{S} = \oint_{\partial S} \vec{F}\cdot d\vec{s}$$



Verify Stokes' Theorem if

$F = (y, -x, x^2+y^2)$

and S is the paraboloid $z = x^2+y^2$ for $z \le 4$

with **normal pointing up and in**.

Remember to check the orientations.

The paraboloid may be parametrized by:

$\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, r^2)$

a. (10) Compute $\iint_S \vec{\nabla}\times\vec{F}\cdot d\vec{S}$ using the following steps:

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ y & -x & x^2+y^2 \end{vmatrix} = i(2y-0) - j(2x-0) + k(-1-1) = (2y, -2x, -2)$$

$\left(\vec{\nabla}\times\vec{F}\right)\left(\vec{R}(r,\theta)\right) = (2r\sin\theta, -2r\cos\theta, -2)$

$\vec{R}_r = (\cos\theta, \sin\theta, 2r)$

$\vec{R}_\theta = (-r\sin\theta, r\cos\theta, 0)$

$\vec{N} = i(-2r^2\cos\theta) - j(2r^2\sin\theta) + k(r\cos^2\theta + r\sin^2\theta) = (-2r^2\cos\theta, -2r^2\sin\theta, r)$

This is oriented correctly as up and in.

$$\iint_S \vec{\nabla}\times\vec{F}\cdot d\vec{S} = \iint \vec{\nabla}\times\vec{F}\cdot\vec{N}\,dr\,d\theta = \int\int(-4r^3\sin\theta\cos\theta + 4r^3\sin\theta\cos\theta - 2r)\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 (-2r)\,dr\,d\theta = 2\pi\left[-r^2\right]_0^2 = -8\pi$$

b. (10) Recall $F = (y, -x, x^2+y^2)$ and S is the paraboloid $z = x^2+y^2$ for $z \le 4$ with **normal pointing up and in**. Compute $\oint_{\partial S} \vec{F}\cdot d\vec{s}$ using the following steps:

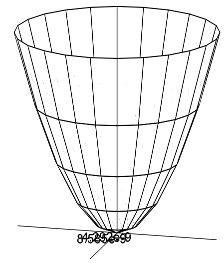
$\vec{r}(\theta) = (2\cos\theta, 2\sin\theta, 4)$

$\vec{v}(\theta) = (-2\sin\theta, 2\cos\theta, 0)$ which is correctly counterclockwise.

$\vec{F}(\vec{r}(\theta)) = (2\sin\theta, -2\cos\theta, 4)$

$$\oint_{\partial S} \vec{F}\cdot d\vec{s} = \int_0^{2\pi} \vec{F}\cdot\vec{v}\,d\theta = \int_0^{2\pi}(-4\sin^2\theta - 4\cos^2\theta)\,d\theta = \int_0^{2\pi}(-4)\,d\theta = -8\pi$$

9. (10 points) The paraboloid at the right is the graph of the equation $z = x^2 + y^2$.



It may be parametrized as

$$\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, r^2).$$

Find the area of the paraboloid for $z \le 4$.

HINT: Use results from #8.

$$\vec{R}_r = (\cos\theta, \sin\theta, 2r)$$

$$\vec{R}_\theta = (-r\sin\theta, r\cos\theta, 0)$$

$$\vec{N} = i(-2r^2\cos\theta) - j(2r^2\sin\theta) + k(r\cos^2\theta + r\sin^2\theta) = (-2r^2\cos\theta, -2r^2\sin\theta, r)$$

$$\left|\vec{N}\right| = \sqrt{4r^4\cos^2\theta + 4r^4\sin^2\theta + r^2} = \sqrt{4r^4 + r^2} = r\sqrt{4r^2+1}$$

$$A = \iint \left|\vec{N}\right| \, dr\, d\theta = \int_0^{2\pi}\int_0^2 r\sqrt{4r^2+1}\, dr\, d\theta = 2\pi\left[\frac{2(4r^2+1)^{3/2}}{3\cdot 8}\right]_0^2 = \frac{\pi}{6}(17^{3/2}-1)$$

10. (10 points) A paraboloid in $\mathbf{R}^4$ with coordinates (w,x,y,z), may be parametrized by

$$(w,x,y,z) = \vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, r^2, r^2) \quad \text{for } 0 \le r \le 3 \text{ and } 0 \le \theta \le 2\pi.$$

Compute $I = \iint (xz\, dw\, dy - wy\, dx\, dz)$ over the surface.

$$w = r\cos\theta, \quad x = r\sin\theta, \quad y = r^2, \quad z = r^2$$

$$\frac{\partial(w,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ 2r & 0 \end{vmatrix} = 2r^2\sin\theta \qquad \frac{\partial(x,z)}{\partial(r,\theta)} = \begin{vmatrix} \sin\theta & r\cos\theta \\ 2r & 0 \end{vmatrix} = -2r^2\cos\theta$$

$$I = \int_0^{2\pi}\int_0^3 (r^3\sin\theta(2r^2\sin\theta) - r^3\cos\theta(-2r^2\cos\theta))\, dr\, d\theta = \int_0^{2\pi}\int_0^3 2r^5\, dr\, d\theta = 2\pi\left[\frac{r^6}{3}\right]_0^3 = 486\pi$$