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Line & Surface Integral Notation

PARAMETRIZED CURVES & LINE INTEGRALS:

Curve:

$$\vec{r}(t) = (x(t), y(t), z(t))$$

Tangent vector:

$$\vec{v} = \frac{d\vec{r}}{dt} = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right)$$

Tangent differential vector:

$$\vec{ds} = \vec{dr} = dx\,\hat{\imath} + dy\,\hat{\jmath} + dz\,\hat{k} = \left(\frac{dx}{dt}\hat{\imath} + \frac{dy}{dt}\hat{\jmath} + \frac{dz}{dt}\hat{k} \right) dt = \vec{v}\,dt = \hat{v}\,|\vec{v}|\,dt = \hat{v}\,ds$$

Tangent differential scalar:

$$ds = |ds| = \sqrt{(dx)^2 + (dy)^2 + (dz)^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\; dt = |\vec{v}|\,dt$$

Arclength integral:

$$s = \int_A^B ds = \int_a^b |\vec{v}|\,dt$$

Integral of a scalar $f(x,y,z)$ along $\vec{r}(t)$ from $A = \vec{r}(a)$ to $B = \vec{r}(b)$:

$$\int_A^B f\,ds = \int_a^b f(\vec{r}(t))\,|\vec{v}|\,dt$$

Total mass:

$$M = \int_A^B \rho\,ds = \int_a^b \rho\,|\vec{v}|\,dt$$

Center of mass:

$$(\bar{x}, \bar{y}, \bar{z}) = \int_A^B (x,y,z)\,\rho\,ds$$

Integral of a vector field $\vec{F} = F_1\,\hat{\imath} + F_2\,\hat{\jmath} + F_3\,\hat{k}$ along $\vec{r}(t)$ from $A = \vec{r}(a)$ to $B = \vec{r}(b)$:

$$\int_A^B \vec{F} \cdot \vec{ds} = \int_A^B (F_1\,dx + F_2\,dy + F_3\,dz) = \int_a^b \left(F_1 \frac{dx}{dt} + F_2 \frac{dy}{dt} + F_3 \frac{dz}{dt} \right) dt = \int_a^b \vec{F} \cdot \vec{v}\,dt = \int_A^B \vec{F} \cdot \hat{v}\,ds$$

SPECIAL FOR CURVES IN R^2:

Normal vector:

$$\vec{n} = \vec{v}^{\perp} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} \\ v_1 & v_2 \end{vmatrix} = v_2\hat{\imath} - v_1\,\hat{\jmath} = \frac{dy}{dt}\hat{\imath} - \frac{dx}{dt}\hat{\jmath} \qquad |\vec{n}| = |\vec{v}|$$

Normal differential vector:

$$\vec{dn} = dy\,\hat{\imath} - dx\,\hat{\jmath} = \left(\frac{dy}{dt}\hat{\imath} - \frac{dx}{dt}\hat{\jmath} \right) dt = \vec{n}\,dt = \hat{n}|\vec{n}|\,dt = \hat{n}\,ds$$

Normal differential scalar:

$$dn = |\vec{dn}| = \sqrt{(dy)^2 + (dx)^2} = ds$$

Integral of the normal component of vector field $\vec{G} = G_1\,\hat{\imath} + G_2\,\hat{\jmath}$ along $\vec{r}(t)$:

$$\int_A^B \vec{G} \cdot \vec{dn} = \int_A^B (G_1 dy - G_2 dx) = \int_a^b \left(G_1 \frac{dy}{dt} - G_2 \frac{dx}{dt} \right) dt = \int_a^b \vec{G} \cdot \vec{n}\,dt = \int_A^B \vec{G} \cdot \hat{n}\,ds$$

Further, if $\vec{G} = \vec{F}^{\perp} = F_2\,\hat{\imath} - F_1\,\hat{\jmath}$ then:

$$\vec{G} \cdot \vec{n} = (F_2, -F_1) \cdot (v_2, -v_1) = \vec{F} \cdot \vec{v}$$

and

$$\int_A^B \vec{G} \cdot \vec{dn} = \int_A^B (F_2\,dy - (-F_1)\,dx) = \int_A^B \vec{F} \cdot \vec{ds}$$

PARAMETRIZED SURFACES & SURFACE INTEGRALS:

Surface:

$$\vec{R}(u,v) = (x(u,v)y(u,v)z(u,v))$$

Tangent vectors:

$$\vec{e}_u = \frac{\partial \vec{R}}{\partial u} = \frac{\partial x}{\partial u}\hat{\imath} + \frac{\partial y}{\partial u}\hat{\jmath} + \frac{\partial z}{\partial u}\hat{k} \quad \text{and} \quad \vec{e}_v = \frac{\partial \vec{R}}{\partial v} = \frac{\partial x}{\partial v}\hat{\imath} + \frac{\partial y}{\partial v}\hat{\jmath} + \frac{\partial z}{\partial v}\hat{k}$$

Normal vector:

$$\vec{N} = \vec{e}_u \times \vec{e}_v = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix} = \frac{\partial(y,z)}{\partial(u,v)}\hat{\imath} + \frac{\partial(z,x)}{\partial(u,v)}\hat{\jmath} + \frac{\partial(x,y)}{\partial(u,v)}\hat{k}$$

Surface differential vector:

$$\vec{dS} = dy\,dz\,\hat{\imath} + dz\,dx\,\hat{\jmath} + dx\,dy\,\hat{k} = \left(\frac{\partial(y,z)}{\partial(u,v)}\hat{\imath} + \frac{\partial(z,x)}{\partial(u,v)}\hat{\jmath} + \frac{\partial(x,y)}{\partial(u,v)}\hat{k}\right)du\,dv$$

$$= \vec{N}\,du\,dv = \hat{N}|\vec{N}|\,du\,dv = \hat{N}\,dS$$

Surface differential scalar:

$$dS = \left|\vec{dS}\right| = \sqrt{(dy\,dz)^2 + (dz\,dx)^2 + (dx\,dy)^2} = \sqrt{\left(\frac{\partial(y,z)}{\partial(u,v)}\right)^2 + \left(\frac{\partial(z,x)}{\partial(u,v)}\right)^2 + \left(\frac{\partial(x,y)}{\partial(u,v)}\right)^2}\,du\,dv$$

$$= \left|\vec{N}\right|\,du\,dv$$

Surface area integral:

$$A = \iint_{\vec{R}} dS = \iint_{\vec{R}} \left|\vec{N}\right|\,du\,dv$$

Integral of a scalar $f(x,y,z)$ over $\vec{R}(u,v)$:

$$\iint_{\vec{R}} f\,dS = \iint_{\vec{R}} f\left(\vec{R}(u,v)\right)\left|\vec{N}\right|\,du\,dv$$

Total mass:

$$M = \iint_{\vec{R}} \rho\,dS = \iint_{\vec{R}} \rho\left|\vec{N}\right|\,du\,dv$$

Center of mass:

$$(\bar{x},\bar{y},\bar{z}) = \iint_{\vec{R}} (x,y,z)\,\rho\,dS$$

Integral of a vector field $\vec{F} = F_1\,\hat{\imath} + F_2\,\hat{\jmath} + F_3\,\hat{k}$ over $\vec{R}(u,v)$:

$$\iint_{\vec{R}} \vec{F}\cdot\vec{dS} = \iint_{\vec{R}} (F_1\,dy\,dz + F_2\,dz\,dx + F_3\,dx\,dy) = \iint_{\vec{R}} \left(F_1\frac{\partial(y,z)}{\partial(u,v)} + F_2\frac{\partial(z,x)}{\partial(u,v)} + F_3\frac{\partial(x,y)}{\partial(u,v)}\right)du\,dv$$

$$= \iint_{\vec{R}} \vec{F}\cdot\vec{N}\,du\,dv = \iint_{\vec{R}} \vec{F}\cdot\hat{N}\,dS$$