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MATH 253 Exam 2 Fall 2006

Sections 201-202 Solutions P. Yasskin

Multiple Choice: (5 points each. No part credit.)

1. Compute $\int_0^2 \int_0^z \int_0^{xz} 15x\,dy\,dx\,dz$.

a. 4

b. 8

c. 16

d. 32 Correct Choice

e. 64

$$\int_0^2 \int_0^z \int_0^{xz} 15x\,dy\,dx\,dz = \int_0^2 \int_0^z \Big[15xy\Big]_{y=0}^{xz} dx\,dz = \int_0^2 \int_0^z 15x^2 z\,dx\,dz = \int_0^2 \Big[5x^3 z\Big]_{x=0}^{z} dz = \int_0^2 5z^4\,dz$$

$$= \Big[z^5\Big]_{z=0}^2 = 32$$

2. Compute $\int_0^2 \int_{y^2}^4 y\sin(x^2)\,dx\,dy$ by interchanging the order of integration.

a. $\dfrac{-\cos 16}{2}$

b. $\dfrac{\cos 16 - 1}{2}$

c. $\dfrac{1-\cos 16}{4}$ Correct Choice

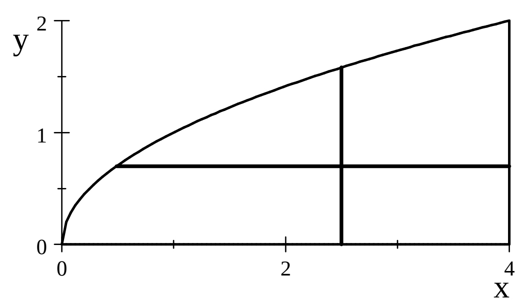
d. $\dfrac{\cos 16}{8}$

e. $\dfrac{\cos 16 - 1}{4}$

Plot the region.

In the original order: For each y between 0 and 2, x runs from y^2 to 4.

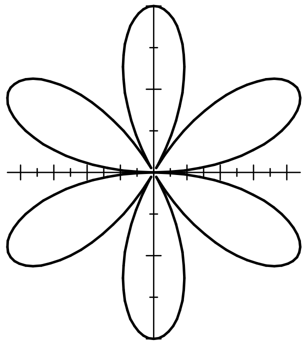
In the reversed order: For each x between 0 and 4, y runs from 0 to $\sqrt{x}$.

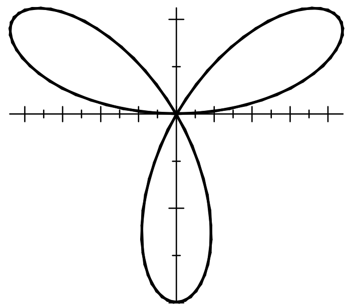


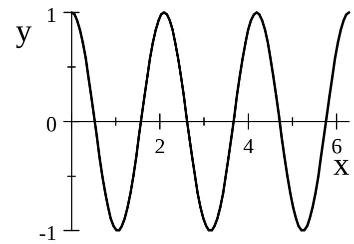
$$\int_0^2 \int_{y^2}^4 y\sin(x^2)\,dx\,dy = \int_0^4 \int_0^{\sqrt{x}} y\sin(x^2)\,dy\,dx = \int_0^4 \left[\frac{y^2}{2}\sin(x^2)\right]_0^{\sqrt{x}} dx = \frac{1}{2}\int_0^4 x\sin(x^2)\,dx$$

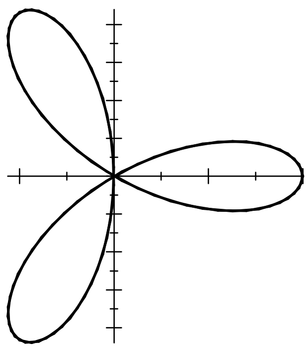
$$= \left[\frac{-\cos(x^2)}{4}\right]_0^4 = \frac{1-\cos 16}{4}$$

3. Which of the following is the polar plot of $r = \cos(3\theta)$?

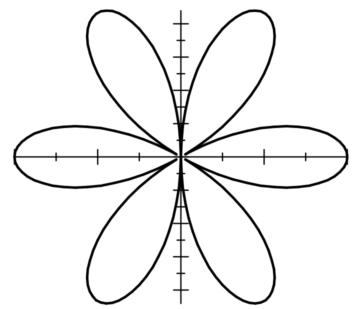
a. 

b. 

c. 

d. 

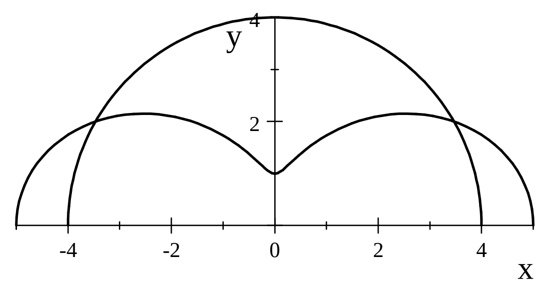
←Correct Choice

e. 

(c) is the rectangular plot of $r = \cos(3\theta)$. (d) is its polar plot because there are 3 positive loops and 3 negative loops which retrace the positive loops.

4. Find the area of the region inside the circle $r = 4$ outside the polar curve $r = 3 + 2\cos(2\theta)$ with $y \geq 0$.

The area is given by the integral:



a. $A = \int_{\pi/3}^{5\pi/3} \int_{3+2\cos(2\theta)}^{4} r\,dr\,d\theta$

b. $A = \int_{\pi/3}^{5\pi/3} \int_{4}^{3+2\cos(2\theta)} dr\,d\theta$

c. $A = \int_{\pi/3}^{5\pi/3} \int_{3+2\cos(2\theta)}^{4} dr\,d\theta$

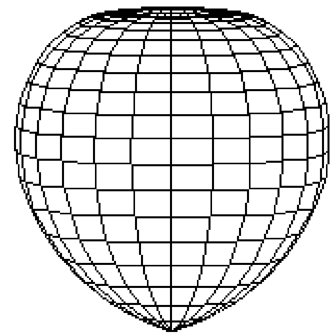
d. $A = \int_{\pi/6}^{5\pi/6} \int_{3+2\cos(2\theta)}^{4} dr\,d\theta$

e. $A = \int_{\pi/6}^{5\pi/6} \int_{3+2\cos(2\theta)}^{4} r\,dr\,d\theta$ Correct Choice

$r = 3 + 2\cos(2\theta) = 4 \quad \Rightarrow \quad \cos(2\theta) = \frac{1}{2} \quad \Rightarrow \quad 2\theta = \frac{\pi}{3} \quad \Rightarrow \quad \theta = \frac{\pi}{6}, \frac{5\pi}{6}$ by symmetry

$A = \iint dA = \int_{\pi/6}^{5\pi/6} \int_{3+2\cos(2\theta)}^{4} r\,dr\,d\theta$

5. Find the volume of the apple given in spherical coordinates by $\rho = 3\varphi$.



The volume is given by the integral:

a. $54\pi \int_0^{2\pi} \varphi^2 \sin\varphi \, d\varphi$

b. $54\pi \int_0^{\pi} \varphi^2 \sin\varphi \, d\varphi$

c. $27\pi \int_0^{2\pi} \varphi^2 \sin\varphi \, d\varphi$

d. $27\pi \int_0^{\pi} \varphi^2 \sin\varphi \, d\varphi$

e. $18\pi \int_0^{\pi} \varphi^3 \sin\varphi \, d\varphi$ Correct Choice

$$V = \iiint dV = \int_0^{2\pi} \int_0^{\pi} \int_0^{3\varphi} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta = 2\pi \int_0^{\pi} \left[\frac{\rho^3}{3} \right]_{\rho=0}^{3\varphi} \sin\varphi \, d\varphi = 18\pi \int_0^{\pi} \varphi^3 \sin\varphi \, d\varphi$$

6. Find a scalar potential f for the vector field $\vec{F} = (y - z, x + z, y - x + 2z)$.
Then evaluate $f(1,1,1) - f(0,0,0)$:

a. 1

b. 2 Correct Choice

c. 4

d. 5

e. 7

$\partial_x f = y - z \qquad f = xy - xz + g(y,z)$

$\partial_y f = x + z \qquad x + \partial_y g = x + z \qquad \partial_y g = z \qquad g = yz + h(z) \qquad f = xy - xz + yz + h(z)$

$\partial_z f = y - x + 2z \qquad -x + y + \dfrac{dh}{dz} = y - x + 2z \qquad \dfrac{dh}{dz} = 2z \qquad h = z^2 \qquad f = xy - xz + yz + z^2$

$f(1,1,1) - f(0,0,0) = (1 - 1 + 1 + 1) - (0 - 0 + 0 + 0) = 2$

7. Which vector field cannot be written as $\vec{\nabla}\times\vec{F}$ for any vector field $\vec{F}$.

a. $\vec{A} = (xz, yz, z^2)$ Correct Choice $\qquad \vec{\nabla}\cdot\vec{A} = z+z+2z = 4z \neq 0$

b. $\vec{B} = (x, y, -2z)$ $\qquad \vec{\nabla}\cdot\vec{B} = 1+1-2 = 0$

c. $\vec{C} = (xz, yz, -z^2)$ $\qquad \vec{\nabla}\cdot\vec{C} = z+z-2z = 0$

d. $\vec{D} = (z\sin x, -yz\cos x, y\sin x)$ $\qquad \vec{\nabla}\cdot\vec{D} = z\cos x - z\cos x = 0$

e. $\vec{E} = (x\sin y, \cos y, x\cos y)$ $\qquad \vec{\nabla}\cdot\vec{E} = \sin y - \sin y = 0$

Since it is always true that $\vec{\nabla}\cdot\vec{\nabla}\times\vec{F} = 0$, $\vec{A}$ cannot be $\vec{A} = \vec{\nabla}\times\vec{F}$.

On the other hand, $\vec{B}$, $\vec{C}$, $\vec{D}$ and $\vec{E}$ probably have vector potentials.

8. Find the total mass of a plate occupying the region between $x = y^2$ and $x = 4$ if the mass density is $\rho = x$.

a. $\dfrac{64}{5}$

b. $\dfrac{128}{5}$ Correct Choice

c. $\dfrac{256}{5}$

d. $\dfrac{32}{3}$

e. $\dfrac{32}{9}$

$x = y^2 = 4 \qquad y = \pm 2$

$$M = \iint \rho\, dA = \int_{-2}^{2}\int_{y^2}^{4} x\,dx\,dy = \int_{-2}^{2}\left[\frac{x^2}{2}\right]_{x=y^2}^{4} dy = \frac{1}{2}\int_{-2}^{2}(16-y^4)\,dy = \frac{1}{2}\left[16y - \frac{y^5}{5}\right]_{y=-2}^{2} = \frac{128}{5}$$

9. Find the center of mass of a plate occupying the region between $x = y^2$ and $x = 4$ if the mass density is $\rho = x$.

a. $\left(\frac{20}{7}, 0\right)$ Correct Choice

b. $\left(\frac{12}{5}, 0\right)$

c. $\left(\frac{512}{7}, 0\right)$

d. $\left(\frac{128}{5}, 0\right)$

e. $\left(\frac{14}{5}, 0\right)$

$M = \dfrac{128}{5}$ By symmetry, $\bar{y} = 0$.

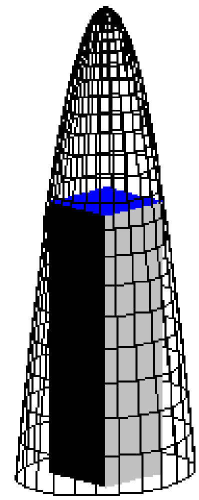
$$x\text{-mom} = M_y = \iint x\rho\, dA = \int_{-2}^{2}\int_{y^2}^{4} x^2\,dx\,dy = \int_{-2}^{2}\left[\frac{x^3}{3}\right]_{x=y^2}^{4} dy = \frac{1}{3}\int_{-2}^{2}(64-y^6)\,dy = \frac{1}{3}\left[64y - \frac{y^7}{7}\right]_{y=-2}^{2}$$

$$= \frac{2}{3}\left(128 - \frac{128}{7}\right) = \frac{512}{7} \qquad \bar{x} = \frac{x\text{-mom}}{M} = \frac{M_y}{M} = \frac{512}{7}\,\frac{5}{128} = \frac{20}{7}$$

Work Out: (12 points each. Part credit possible. Show all work.)

10. Find the dimensions and volume of the largest box which sits on the xy-plane and whose upper vertices are on the elliptic paraboloid $z = 12 - 2x^2 - 3y^2$.

You do not need to show it is a maximum.
You MUST eliminate the constraint.
Do not use Lagrange multipliers.



Maximize $V = (2x)(2y)z = 4xy(12 - 2x^2 - 3y^2) = 48xy - 8x^3y - 12xy^3$

$V_x = 48y - 24x^2y - 12y^3 = -12y(2x^2 + y^2 - 4) = 0$

$V_y = 48x - 8x^3 - 36xy^2 = -4x(2x^2 + 9y^2 - 12) = 0$

Since $x = 0$ or $y = 0$ gives $V = 0$, we can assume $x \neq 0$ and $y \neq 0$.

$2x^2 + y^2 = 4$ and $2x^2 + 9y^2 = 12$

Subtract: $8y^2 = 8 \quad \Rightarrow \quad y = 1$

Substitute back: $2x^2 + 1 = 4 \quad \Rightarrow \quad x^2 = \frac{3}{2} \quad \Rightarrow \quad x = \sqrt{\frac{3}{2}}$

Substitute back: $z = 12 - 2x^2 - 3y^2 = 12 - 2\left(\frac{3}{2}\right) - 3(1) = 6$

The dimensions are: $L = 2x = \sqrt{6} \quad W = 2y = 2 \quad H = z = 6$

The volume is: $V = LWH = \sqrt{6}(2)(6) = 12\sqrt{6}$

11. A pot of water is sitting on a stove. The pot is a cylinder of radius 3 inches and height 4 inches. If the origin is located at the center of the bottom, then the temperature of the water is $T = 102 + x^2 + y^2 - z$. Find the average temperature of the water: $T_{\text{ave}} = \dfrac{\iiint T\,dV}{\iiint dV}$.

$$\iiint dV = \int_0^{2\pi}\int_0^3\int_0^4 r\,dz\,dr\,d\theta = \Big[\theta\Big]_0^{2\pi}\left[\frac{r^2}{2}\right]_0^3\Big[z\Big]_0^4 = 2\pi\left(\frac{9}{2}\right)(4) = 36\pi \qquad (= \pi r^2 h)$$

$$\iiint T\,dV = \int_0^{2\pi}\int_0^3\int_0^4 (102 + r^2 - z)r\,dz\,dr\,d\theta = 2\pi\int_0^3\left[102z + r^2z - \frac{z^2}{2}\right]_{z=0}^4 r\,dr$$

$$= 2\pi\int_0^3 (408 + 4r^2 - 8)r\,dr = 2\pi\int_0^3 (400r + 4r^3)\,dr = 2\pi[200r^2 + r^4]_{r=0}^3 = 2\pi(1881) = 3762\pi$$

$$T_{\text{ave}} = \frac{3762\pi}{36\pi} = \frac{209}{2}$$

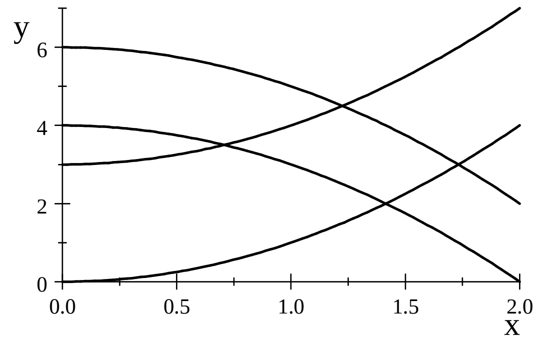
12. Compute $\iint_D x\,dx\,dy$ over the "diamond" shaped region bounded by the curves

$$y = x^2 \qquad y = 3 + x^2 \qquad y = 4 - x^2 \qquad y = 6 - x^2$$

Use the curvilinear coordinates

$$u = y + x^2 \quad \text{and} \quad v = y - x^2.$$

(Half credit for using rectangular coordinates.)



The boundary curves are $v = 0$, $v = 3$, $u = 4$, $u = 6$

We solve for x and y and compute the Jacobian factor:

$$u + v = 2y \qquad u - v = 2x^2 \qquad x = \frac{\sqrt{u-v}}{\sqrt{2}} \qquad y = \frac{u+v}{2}$$

$$J = \left\| \begin{matrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{matrix} \right\| = \left\| \begin{matrix} \frac{(u-v)^{-1/2}}{2\sqrt{2}} & \frac{1}{2} \\ -\frac{(u-v)^{-1/2}}{2\sqrt{2}} & \frac{1}{2} \end{matrix} \right\| = \left| \frac{(u-v)^{-1/2}}{4\sqrt{2}} + \frac{(u-v)^{-1/2}}{4\sqrt{2}} \right| = \frac{(u-v)^{-1/2}}{2\sqrt{2}}$$

The integrand is $x = \frac{\sqrt{u-v}}{\sqrt{2}}$. So

$$\iint_D x\,dx\,dy = \int_0^3 \int_4^6 \frac{\sqrt{u-v}}{\sqrt{2}} \frac{(u-v)^{-1/2}}{2\sqrt{2}}\,du\,dv = \int_0^3 \int_4^6 \frac{1}{4}\,du\,dv = \frac{1}{4}(3)(2) = \frac{3}{2}$$

13. The sides of a cylinder C of radius 3 and height 4 may be parametrized by

$R(h,\theta) = (3\cos\theta, 3\sin\theta, h)$ for $0 \le \theta \le 2\pi$ and $0 \le h \le 4$.

Compute $\iint_C \vec{\nabla} \times \vec{F} \bullet d\vec{S}$ for $\vec{F} = (-yz^2, xz^2, z^3)$ and outward normal.

HINT: Find $\vec{e}_h$, $\vec{e}_\theta$, $\vec{N} = \vec{e}_h \times \vec{e}_\theta$, $\vec{\nabla} \times \vec{F}$ and $\left(\vec{\nabla} \times \vec{F}\right)\left(\vec{R}(h,\theta)\right)$.

$R(h,\theta) = (3\cos\theta, 3\sin\theta, h)$

$$\begin{array}{lcccc} & \hat{\imath} & \hat{\jmath} & \hat{k} & \\ \vec{e}_h = (& 0, & 0, & 1 &) \\ \vec{e}_\theta = (& -3\sin\theta, & 3\cos\theta, & 0 &) \end{array}$$

$\vec{N} = \vec{e}_h \times \vec{e}_\theta = \hat{\imath}(-3\cos\theta) - \hat{\jmath}(3\sin\theta) + \hat{k}(0) = (-3\cos\theta, -3\sin\theta, 0)$

Reverse $\vec{N} = (3\cos\theta, 3\sin\theta, 0)$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ -yz^2 & xz^2 & z^3 \end{vmatrix} = \hat{\imath}(0 - 2xz) - \hat{\jmath}(0 - -2yz) + \hat{k}(z^2 - -z^2) = (-2xz, -2yz, 2z^2)$$

$$\left(\vec{\nabla} \times \vec{F}\right)\left(\vec{R}(h,\theta)\right) = (-6h\cos\theta, -6h\sin\theta, 2h^2)$$

$$\vec{\nabla} \times \vec{F} \cdot \vec{N} = -18h\cos^2\theta - 18h\sin^2\theta = -18h$$

$$\iint_C \vec{\nabla} \times \vec{F} \cdot d\vec{S} = \int\int \vec{\nabla} \times \vec{F} \cdot \vec{N}\, dh\, d\theta = \int_0^{2\pi}\int_0^4 -18h\, dh\, d\theta = -[2\pi 9h^2]_0^4 = -288\pi$$

14. The hemispherical surface $x^2 + y^2 + z^2 = 9$ has surface density $\rho = x^2 + y^2$.

The surface may be parametrized by $\vec{R}(\varphi,\theta) = (3\sin\varphi\cos\theta, 3\sin\varphi\sin\theta, 3\cos\varphi)$.

Find the mass and center of mass of the surface.

HINT: Find $\vec{e}_\varphi$, $\vec{e}_\theta$, $\vec{N} = \vec{e}_\varphi \times \vec{e}_\theta$, $\left|\vec{N}\right|$ and $\rho\left(\vec{R}(\varphi,\theta)\right)$.

$$\vec{R}(\varphi,\theta) = (3\sin\varphi\cos\theta, 3\sin\varphi\sin\theta, 3\cos\varphi)$$

$$\begin{matrix} & \hat{\imath} & \hat{\jmath} & \hat{k} \\ \vec{e}_\varphi = & (3\cos\varphi\cos\theta, & 3\cos\varphi\sin\theta, & -3\sin\varphi) \\ \vec{e}_\theta = & (-3\sin\varphi\sin\theta, & 3\sin\varphi\cos\theta, & 0 \quad) \end{matrix}$$

$$\vec{N} = \vec{e}_\varphi \times \vec{e}_\theta = \hat{\imath}(9\sin^2\varphi\cos\theta) - \hat{\jmath}(-9\sin^2\varphi\sin\theta) + \hat{k}(9\sin\varphi\cos\varphi\cos^2\theta + 9\sin\varphi\cos\varphi\sin^2\theta)$$

$$\vec{N} = (9\sin^2\varphi\cos\theta, 9\sin^2\varphi\sin\theta, 9\sin\varphi\cos\varphi)$$

$$\left|\vec{N}\right| = \sqrt{(9\sin^2\varphi\cos\theta)^2 + (9\sin^2\varphi\sin\theta)^2 + (9\sin\varphi\cos\varphi)^2} = 9\sqrt{\sin^4\varphi + \sin^2\varphi\cos^2\varphi} = 9\sin\varphi$$

$$\rho\left(\vec{R}(\varphi,\theta)\right) = (3\sin\varphi\cos\theta)^2 + (3\sin\varphi\sin\theta)^2 = 9\sin^2\varphi$$

$$M = \iint \rho\, dS = \iint \rho\left(\vec{R}(\varphi,\theta)\right)\left|\vec{N}\right| d\varphi\, d\theta = \int_0^{2\pi}\int_0^{\pi/2} 9\sin^2\varphi 9\sin\varphi\, d\varphi\, d\theta = 2\pi 81\int_0^{\pi/2}(1 - \cos^2\varphi)\sin\varphi\, d\varphi$$

Let $\quad u = \cos\varphi \qquad du = -\sin\varphi\, d\varphi$

$$M = -162\pi\int_1^0 (1 - u^2)\, du = -162\pi\left[u - \frac{u^3}{3}\right]_1^0 = 162\pi\left(1 - \frac{1}{3}\right) = \frac{324\pi}{3} = 108\pi$$

By symmetry $\bar{x} = \bar{y} = 0$

$$z\text{-mom} = M_{xy} = \iint z\rho\, dS = \iint 3\cos\varphi\,\rho\left(\vec{R}(\varphi,\theta)\right)\left|\vec{N}\right| d\varphi\, d\theta = \int_0^{2\pi}\int_0^{\pi/2} 3\cos\varphi 9\sin^2\varphi 9\sin\varphi\, d\varphi\, d\theta$$

$$= 2\pi 243\int_0^{\pi/2}\cos\varphi\sin^3\varphi\, d\varphi = 486\pi\left[\frac{\sin^4\varphi}{4}\right]_0^{\pi/2} = \frac{243\pi}{2}$$

$$\bar{z} = \frac{z\text{-mom}}{M} = \frac{243\pi}{2}\frac{1}{108\pi} = \frac{9}{8}$$