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MATH 253 Final Exam Fall 2006

Sections 201,202 Solutions P. Yasskin

1-10	/50
11	/15
12	/15
13	/15
14	/15
Total	/110

Multiple Choice: (5 points each. No part credit.)

1. For the curve $\vec{r}(t) = (t\cos t, t\sin t)$, which of the following is false?

a. The velocity is $\vec{v} = (\cos t - t\sin t, \sin t + t\cos t)$

b. The speed is $|\vec{v}| = \sqrt{1+t^2}$

c. The acceleration is $\vec{a} = (-2\sin t - t\cos t, 2\cos t - t\sin t)$

d. The arclength between $t = 0$ and $t = 1$ is $L = \int_0^1 t\sqrt{1+t^2}\,dt$ Correct Choice

e. The tangential acceleration is $a_T = \dfrac{t}{\sqrt{1+t^2}}$

$\vec{v} = (\cos t - t\sin t, \sin t + t\cos t)$

$|\vec{v}| = \sqrt{(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2} = \sqrt{\cos^2 t + t^2\cos^2 t + \sin^2 t + t^2\sin^2 t} = \sqrt{1+t^2}$

$\vec{a} = (-2\sin t - t\cos t, 2\cos t - t\sin t)$

$L = \int_0^1 |\vec{v}|\,dt = \int_0^1 \sqrt{1+t^2}\,dt$

$a_T = \dfrac{d|\vec{v}|}{dt} = \dfrac{2t}{2\sqrt{1+t^2}}$ or

$a_T = \vec{a}\cdot\hat{T} = (-2\sin t - t\cos t, 2\cos t - t\sin t)\cdot\dfrac{1}{\sqrt{1+t^2}}(\cos t - t\sin t, \sin t + t\cos t)$

$= \dfrac{1}{\sqrt{1+t^2}}[(-2\sin t - t\cos t)(\cos t - t\sin t) + (2\cos t - t\sin t)(\sin t + t\cos t)] = \dfrac{t}{\sqrt{1+t^2}}$

2. Find the line perpendicular to the surface $x^2z^2 + y^4 = 5$ at the point $(2,1,1)$.

a. $(x,y,z) = (1+t, 1+t, 2+2t)$

b. $(x,y,z) = (1+2t, 1+t, 2+t)$

c. $(x,y,z) = (2+t, 1+t, 1+2t)$ Correct Choice

d. $(x,y,z) = (1+2t, 1+t, 2+2t)$

e. $(x,y,z) = (2+2t, 1+t, 1+1t)$

$f = x^2z^2 + y^4$ $P = (2,1,1)$ $\vec{\nabla} f = (2xz^2, 4y^3, 2x^2z)$ $\vec{\nabla} f\big|_P = (4,4,8)$ $\vec{v} = (1,1,2)$

$X = P + t\vec{v}$ $(x,y,z) = (2,1,1) + t(1,1,2) = (2+t, 1+t, 1+2t)$

3. Let $L = \lim_{(x,y)\to(0,0)} \dfrac{e^{\left(x^2+y^2\right)} - 1}{x^2 + y^2}$

a. L exists and $L = 1$ by looking at the paths $y = mx$.

b. L does not exist by looking at the paths $y = x$ and $y = -x$.

c. L does not exist by looking at polar coordinates.

d. L exists and $L = 0$ by looking at polar coordinates.

e. L exists and $L = 1$ by looking at polar coordinates. Correct Choice

Along $y = mx$, we have $L = \lim_{x\to 0} \dfrac{e^{\left(1+m^2\right)x^2} - 1}{(1+m^2)x^2} \overset{\text{l'H}}{=} \lim_{x\to 0} \dfrac{e^{\left(1+m^2\right)x^2}(1+m^2)2x}{(1+m^2)2x} = 1$,

for all m including 1 and -1 which proves nothing.

In polar coordinates, $L = \lim_{r\to 0} \dfrac{e^{r^2} - 1}{r^2} \overset{\text{l'H}}{=} \lim_{r\to 0} \dfrac{e^{r^2} 2r}{2r} = 1$, which proves the limit exists and $= 1$.

4. The point $(1,-3)$ is a critical point of the function $f = xy^2 - 3x^3 + 6y$. It is a

a. local minimum.

b. local maximum.

c. saddle point. Correct Choice

d. inflection point.

e. The Second Derivative Test fails.

$f_x = y^2 - 9x^2$ $\quad f_y = 2xy + 6$ $\quad f_{xx} = -18x$ $\quad f_{yy} = 2x$ $\quad f_{xy} = 2y$

$f_{xx}(1,-3) = -18$ $\quad f_{yy}(1,-3) = 2$ $\quad f_{xy}(1,-3) = -6$ $\quad D = f_{xx}f_{yy} - f_{xy}^2 = -36 - 36 = -72$

saddle point

5. Compute the line integral $\int \vec{F} \cdot d\vec{s}$ for the vector field $\vec{F} = (y, x + 2y)$ along the curve $\vec{r}(t) = \left(e^{\sin\left(t^2\right)}, e^{\cos\left(t^2\right)}\right)$ for $0 \le t \le \sqrt{\pi}$. (HINT: Find a scalar potential.)

a. $e^2 + e - \dfrac{1}{e} - \dfrac{1}{e^2}$

b. $\dfrac{1}{e^2} + \dfrac{1}{e} - e - e^2$ Correct Choice

c. $e^2 - e + \dfrac{1}{e} - \dfrac{1}{e^2}$

d. $\dfrac{1}{e^2} - \dfrac{1}{e} + e - e^2$

e. 0

$\vec{F} = \vec{\nabla} f$ for $f = xy + y^2$ $\quad A = \vec{r}(0) = (e^{\sin 0}, e^{\cos 0}) = (1, e)$ $\quad B = \vec{r}(\sqrt{\pi}\,) = (e^{\sin\pi}, e^{\cos\pi}) = (1, e^{-1})$

By the F.T.C.C.

$\int_A^B \vec{F} \cdot d\vec{s} = \int_A^B \vec{\nabla} f \cdot d\vec{s} = f(B) - f(A) = f(1, e^{-1}) - f(1, e) = (e^{-1} + e^{-2}) - (e + e^2) = \dfrac{1}{e^2} + \dfrac{1}{e} - e - e^2$

6. Compute the line integral $\int y\,dx - x\,dy$ along the curve $y = x^2$ from $(-3,9)$ to $(0,0)$.
HINT: The curve may be parametrized as $r(t) = (t,t^2)$.

a. -9 Correct Choice
b. -3
c. 1
d. 3
e. 9

$r(t) = (t,t^2) \qquad \vec{v} = (1,2t)$ Orientation OK.

$\vec{F} = (y,-x) = (t^2,-t) \qquad \vec{F}\cdot\vec{v} = t^2 - 2t^2 = -t^2$

$$\int y\,dx - x\,dy = \int \vec{F}\cdot d\vec{s} = \int \vec{F}\cdot\vec{v}\,dt = \int_{-3}^{0} -t^2\,d\theta = \left[-\frac{t^3}{3}\right]_{-3}^{0} = 0 - \left(-\frac{-27}{3}\right) = -9$$

7. Consider the quarter cylinder surface $x^2 + y^2 = 4$ with $x \ge 0$, $y \ge 0$ and $0 \le z \le 8$.
Find the total mass of the quarter cylinder surface if the density is $\rho = x$.
The surface may be parametrized by $\vec{R}(\theta,h) = (2\cos\theta, 2\sin\theta, h)$.

a. 32 Correct Choice
b. 32π
c. 8
d. 8π
e. 64π

$\vec{e}_\theta = (-2\sin\theta, 2\cos\theta, \ 0\ ) \qquad \vec{N} = (2\cos\theta, 2\sin\theta, 0)$

$\vec{e}_h = (\ 0\ ,\ 0\ ,\ 1\ ) \qquad \left|\vec{N}\right| = \sqrt{4\cos^2\theta + 4\sin^2\theta} = 2$

$$M = \iint \rho\,dS = \int_0^8\int_0^{\pi/2} x\left|\vec{N}\right|\,d\theta\,dh = \int_0^8\int_0^{\pi/2} 2\cos\theta\,2\,d\theta\,dh = 4(8)\Big[\sin\theta\Big]_0^{\pi/2} = 32$$

8. Consider the quarter cylinder surface $x^2 + y^2 = 4$ with $x \ge 0$, $y \ge 0$ and $0 \le z \le 8$.
Find the y-component of the center of mass of the quarter cylinder if the density is $\rho = x$.

a. $\frac{4}{\pi}$
b. $\frac{\pi}{4}$
c. 32
d. 2
e. 1 Correct Choice

$$y\text{-mom} = \iint y\rho\,dS = \int_0^8\int_0^{\pi/2} yx\left|\vec{N}\right|\,d\theta\,dh = \int_0^8\int_0^{\pi/2} 4\sin\theta\cos\theta\,2\,d\theta\,dh = 8(8)\left[\frac{\sin^2\theta}{2}\right]_0^{\pi/2} = 32$$

$$\bar{y} = \frac{y\text{-mom}}{M} = \frac{32}{32} = 1$$

9. Compute the line integral $\oint x^2 y\,dx - xy^2\,dy$ counterclockwise around the circle $x^2 + y^2 = 16$. (HINT: Use a theorem.)

a. -128π Correct Choice

b. -64π

c. 0

d. 64π

e. 128π

Use Green's Theorem:

$$\oint_{\partial R} P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy \quad \text{with} \quad P = x^2 y \quad \text{and} \quad Q = -xy^2.$$

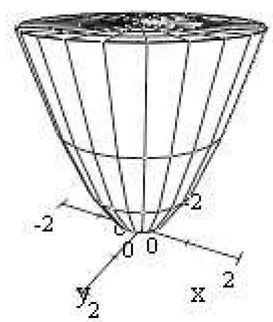
$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = -y^2 - x^2 = -r^2 \qquad dx\,dy = r\,dr\,d\theta$$

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy = -\int_0^{2\pi} \int_0^4 r^2 r\,dr\,d\theta = -2\pi \left[\frac{r^4}{4} \right]_{r=0}^4 = -128\pi$$

10. Consider the parabolic surface P given by $z = x^2 + y^2$ for $z \le 4$ with normal pointing up and in, the disk D given by $x^2 + y^2 \le 4$ and $z = 4$ with normal pointing up, and the volume V between them. Given that for a certain vector field $\vec{F}$ we have

$$\iiint_V \vec{\nabla} \cdot \vec{F}\,dV = 13 \quad \text{and} \quad \iint_D \vec{F} \cdot d\vec{S} = 4$$

compute $\iint_P \vec{F} \cdot d\vec{S}$.



a. -17

b. -9 Correct Choice

c. 5

d. 9

e. 17

By Gauss' Theorem: $$\iiint_V \vec{\nabla} \cdot \vec{F}\,dV = \iint_D \vec{F} \cdot d\vec{S} - \iint_P \vec{F} \cdot d\vec{S}$$

The minus sign reverses the orientation of P to point outward. Thus

$$\iint_P \vec{F} \cdot d\vec{S} = \iint_D \vec{F} \cdot d\vec{S} - \iiint_V \vec{\nabla} \cdot F\,dV = 4 - 13 = -9$$

Work Out: (15 points each. Part credit possible.)

11. Find the point in the first octant on the graph of $xy^2z^4 = 32$ which is closest to the origin.

You do not need to show it is a maximum. You MUST use the Method of Lagrange Multipliers.

Half credit for the Method of Elminating the Constraint.

Minimize $f = x^2 + y^2 + z^2$ subject to $g = xy^2z^4 = 32$.

Method 1: Lagrange Multipliers:

$\vec{\nabla} f = (2x, 2y, 2z) \qquad \vec{\nabla} g = (y^2z^4, 2xyz^4, 4xy^2z^3)$

$\vec{\nabla} f = \lambda \vec{\nabla} g \quad \Rightarrow \quad 2x = \lambda y^2z^4, \qquad 2y = \lambda 2xyz^4, \qquad 2z = \lambda 4xy^2z^3$

x, y and z cannot be 0 to satisfy the constraint.

$$\lambda = \frac{2x}{y^2z^4} = \frac{1}{xz^4} = \frac{1}{2xy^2z^2} \quad \Rightarrow \quad 2x^2 = y^2, \qquad 4x^2 = z^2 \quad \Rightarrow \quad y = \sqrt{2}\,x, \qquad z = 2x$$

$$32 = xy^2z^4 = x\left(\sqrt{2}\,x\right)^2(2x)^4 = 32x^7 \quad \Rightarrow \quad x = 1 \qquad y = \sqrt{2} \qquad z = 2$$

Method 2: Eliminate the Constraint:

$$x = \frac{32}{y^2z^4} \qquad f = \frac{2^{10}}{y^4z^8} + y^2 + z^2$$

$$f_y = -\frac{2^{12}}{y^5z^8} + 2y = 0 \qquad f_z = -\frac{2^{13}}{y^4z^9} + 2z = 0 \quad \Rightarrow \quad y^6z^8 = 2^{11} \qquad y^4z^{10} = 2^{12}$$

$$\Rightarrow \quad 2 = \frac{y^4z^{10}}{y^6z^8} = \frac{z^2}{y^2} \quad \Rightarrow \quad z = \sqrt{2}\,y \quad \Rightarrow \quad y^6\left(\sqrt{2}\,y\right)^8 = 2^{11} \quad \Rightarrow \quad y^{14} = 2^7$$

$$\Rightarrow \quad y = \sqrt{2} \qquad z = 2 \qquad x = \frac{32}{y^2z^4} = \frac{2^5}{2 \cdot 2^4} = 1$$

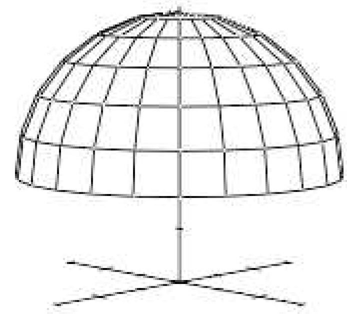
12. The hemisphere H given by

$$x^2 + y^2 + (z-2)^2 = 9 \quad \text{for} \quad z \ge 2$$

has center $(0,0,2)$ and radius 3. Verify Stokes' Theorem

$$\iint_H \vec{\nabla} \times \vec{F} \cdot d\vec{S} = \oint_{\partial H} \vec{F} \cdot d\vec{s}$$

for this hemisphere H with normal pointing up and out and the vector field $\vec{F} = (yz, -xz, z)$.



Be sure to check and explain the orientations. Use the following steps:

a. The hemisphere may be parametrized by

$$\vec{R}(\theta,\varphi) = (3\sin\varphi\cos\theta, 3\sin\varphi\sin\theta, 2 + 3\cos\varphi)$$

Compute the surface integral by successively finding:

$$\vec{e}_\theta, \quad \vec{e}_\varphi, \quad \vec{N}, \quad \vec{\nabla} \times \vec{F}, \quad \vec{\nabla} \times \vec{F}\left(\vec{R}(\theta,\varphi)\right), \quad \iint_H \vec{\nabla} \times \vec{F} \cdot d\vec{S}$$

$$\begin{vmatrix} & \hat{\imath} & \hat{\jmath} & \hat{k} \\ \vec{e}_\theta = & (-3\sin\varphi\sin\theta, & 3\sin\varphi\cos\theta, & 0\ ) \\ \vec{e}_\varphi = & (3\cos\varphi\cos\theta, & 3\cos\varphi\sin\theta, & -3\sin\varphi) \end{vmatrix}$$

$$\vec{N} = \vec{e}_\theta \times \vec{e}_\varphi = \hat{\imath}(-9\sin^2\varphi\cos\theta) - \hat{\jmath}(9\sin^2\varphi\sin\theta) + \hat{k}(-9\sin\varphi\cos\varphi\sin^2\theta - 9\sin\varphi\cos\varphi\cos^2\theta)$$

$$= (-9\sin^2\varphi\cos\theta, -9\sin^2\varphi\sin\theta, -9\sin\varphi\cos\varphi)$$

$\vec{N}$ points down and in. Reverse it: $\vec{N} = (9\sin^2\varphi\cos\theta, 9\sin^2\varphi\sin\theta, 9\sin\varphi\cos\varphi)$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ yz, & -xz, & z \end{vmatrix} = \hat{\imath}(0 - -x) - \hat{\jmath}(0 - y) + \hat{k}(-z - z) = (x, y, -2z)$$

$$\vec{\nabla} \times \vec{F}\left(\vec{R}(r,\theta)\right) = (3\sin\varphi\cos\theta, 3\sin\varphi\sin\theta, -2(2 + 3\cos\varphi))$$

$$\vec{\nabla} \times \vec{F} \cdot \vec{N} = 27\sin^3\varphi\cos^2\theta + 27\sin^3\varphi\sin^2\theta - 18\sin\varphi\cos\varphi(2 + 3\cos\varphi)$$

$$= 27\sin^3\varphi - 36\sin\varphi\cos\varphi - 54\sin\varphi\cos^2\varphi$$

$$\iint_H \vec{\nabla} \times \vec{F} \cdot d\vec{S} = \iint_H \vec{\nabla} \times \vec{F} \cdot \vec{N}\, d\theta\, d\varphi = \int_0^{\pi/2}\int_0^{2\pi} (27\sin^3\varphi - 36\sin\varphi\cos\varphi - 54\sin\varphi\cos^2\varphi)\, d\theta\, d\varphi$$

$$= 2\pi\int_0^{\pi/2} (27(1-\cos^2\varphi)\sin\varphi - 36\sin\varphi\cos\varphi - 54\sin\varphi\cos^2\varphi)\, d\varphi \qquad \text{Let } u = \cos\varphi.$$

$$= 2\pi\left[-27\left(\cos\varphi - \frac{\cos^3\varphi}{3}\right) + 18\cos^2\varphi + 18\cos^3\varphi\right]_0^{\pi/2} = -2\pi\left(-27\left(1 - \frac{1}{3}\right) + 18 + 18\right)$$

$$= -36\pi$$

Problem Continued

b. Parametrize the boundary circle ∂H and compute the line integral by successively finding:

$\vec{r}(\theta)$, $\vec{v}(\theta)$, $\vec{F}(\vec{r}(\theta))$, $\oint_{\partial H} \vec{F} \cdot d\vec{s}$. Recall: $\vec{F} = (yz, -xz, z)$

$\vec{r}(\theta) = (3\cos\theta, 3\sin\theta, 2)$

$\vec{v}(\theta) = (-3\sin\theta, 3\cos\theta, 0)$

By the right hand rule the upper curve must be traversed counterclockwise which $\vec{v}$ does.

$\vec{F}(\vec{r}(\theta)) = (6\sin\theta, -6\cos\theta, 2)$

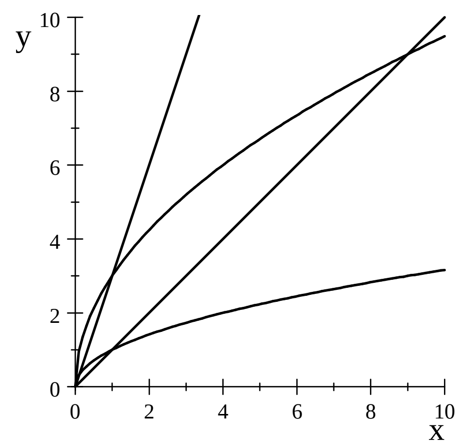
$$\oint_{\partial C} \vec{F} \cdot d\vec{s} = \int_0^{2\pi} \vec{F} \cdot \vec{v}\, d\theta = \int_0^{2\pi} -18\sin^2\theta - 18\cos^2\theta\, d\theta = \int_0^{2\pi} -18\, d\theta = -36\pi$$

They agree!

13. Compute $\iint \frac{1}{x^2}\, dx\, dy$ over the diamond shaped region bounded by the curves

$y = \sqrt{x}$, $y = 3\sqrt{x}$, $y = x$ and $y = 3x$.

HINT: Let $u = \frac{y^2}{x}$ and $v = \frac{y}{x}$.



We solve for x and y so we can compute the Jacobian:

$\frac{u}{v} = \frac{y^2}{x}\frac{x}{y} = y \qquad x = \frac{y}{v} = \frac{u}{v^2}$ So $x = \frac{u}{v^2} \qquad y = \frac{u}{v}$

$$J = \left\| \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \right\| = \left\| \begin{vmatrix} \frac{1}{v^2} & \frac{-2u}{v^3} \\ \frac{1}{v} & \frac{-u}{v^2} \end{vmatrix} \right\| = \left| \frac{-u}{v^4} - \frac{-2u}{v^4} \right| = \frac{u}{v^4}$$

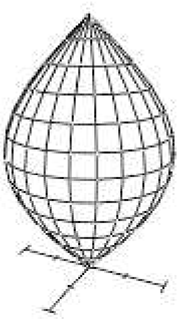
The boundaries are: $y^2 = x$ or $u = 1$. $\quad y^2 = 9x$ or $u = 9$.
$y = x$ or $v = 1$. $\quad y = 3x$ or $v = 3$.

The integrand is: $\frac{1}{x^2} = \frac{v^4}{u^2}$ So

$$\iint \frac{1}{x^2}\, dx\, dy = \int_1^3 \int_1^9 \frac{v^4}{u^2} \cdot \frac{u}{v^4}\, du\, dv = \int_1^3 dv \int_1^9 \frac{1}{u}\, du = \Big[v\Big]_1^3 \Big[\ln|u|\Big]_1^9 = [3-1][\ln 9 - \ln 1] = 2\ln 9$$

14. The surface of a football may be approximated in cylindrical coordinates by

$$r = \sin z \qquad \text{for} \quad 0 \le z \le \pi$$



Verify Gauss' Theorem $\displaystyle \iiint_V \vec{\nabla} \cdot \vec{F}\, dV = \iint_{\partial V} \vec{F} \cdot d\vec{S}$

for the volume inside the football and the vector field

$$\vec{F} = (2x, 2y, x^2 + y^2)$$

Use the following steps:

a. Compute the volume integral by computing $\vec{\nabla} \cdot \vec{F}$ in rectangular coordinates and then $\displaystyle \iiint_V \vec{\nabla} \cdot \vec{F}\, dV$ in cylindrical coordinates.

$$\vec{\nabla} \cdot \vec{F} = 2 + 2 + 0 = 4$$

$$\iiint \vec{\nabla} \cdot \vec{F}\, dV = \int_0^{2\pi} \int_0^{\pi} \int_0^{\sin z} 4r\, dr\, dz\, d\theta = 2\pi \int_0^{\pi} \left[2r^2 \right]_{r=0}^{\sin z} dz = 2\pi \int_0^{\pi} 2\sin^2 z\, dz$$

$$= 2\pi \int_0^{\pi} 1 - \cos 2z\, dz = 2\pi \left[z - \frac{\sin 2z}{2} \right]_0^{\pi} = 2\pi^2$$

b. The surface of the football may be parametrized by $\vec{R}(\theta, h) = (\sin h \cos\theta, \sin h \sin\theta, h)$.

Compute the surface integral by successively finding
$\vec{e}_\theta$, $\vec{e}_h$, $\vec{N}$, $\vec{F}\left(\vec{R}(\theta,h)\right)$, $\vec{F} \cdot \vec{N}$, and $\iint \vec{F} \cdot d\vec{S}$.

$$\vec{e}_\theta = (-\sin h \sin\theta, \sin h \cos\theta, 0)$$

$$\vec{e}_h = (\cos h \cos\theta, \cos h \sin\theta, 1)$$

$$\vec{N} = \vec{e}_\theta \times \vec{e}_h = \hat{\imath}(\sin h \cos\theta) - \hat{\jmath}(-\sin h \sin\theta) + \hat{k}(-\sin h \cos h \sin^2\theta - \sin h \cos h \cos^2\theta)$$

$$= (\sin h \cos\theta, \sin h \sin\theta, -\sin h \cos h)$$

$$\vec{F}\left(\vec{R}(\theta,h)\right) = (2\sin h \cos\theta, 2\sin h \sin\theta, \sin^2 h)$$

$$\vec{F} \cdot \vec{N} = 2\sin^2 h \cos^2\theta + 2\sin^2 h \sin^2\theta - \sin^3 h \cos h = 2\sin^2 h - \sin^3 h \cos h$$

$$\iint \vec{F} \cdot d\vec{S} = \int_0^{2\pi} \int_0^{\pi} \vec{F} \cdot \vec{N}\, dh\, d\theta = \int_0^{2\pi} \int_0^{\pi} (2\sin^2 h - \sin^3 h \cos h)\, dh\, d\theta$$

$$= 2\pi \int_0^{\pi} (1 - \cos 2h - \sin^3 h \cos h)\, dh = 2\pi \left[h - \frac{\sin 2h}{2} - \frac{\sin^4 h}{4} \right]_0^{\pi} = 2\pi^2$$