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|---|---|
| 1-14 | /70 |
| 15 | /10 |
| 16 | /10 |
| 17 | /10 |
| Total | /100 |

Name__________________________

MATH 251/253 (circle one) Exam 1 Fall 2014

Sections 508/201/202(circle one) Solutions P. Yasskin

Multiple Choice: (5 points each. No part credit.)

**1.** The vertices of a triangle are $A = \left(2, 1, \sqrt{2}\right)$, $B = \left(3, 2, 2\sqrt{2}\right)$ and $C = \left(4, 3, \sqrt{2}\right)$. Find the angle at $A$.

**a.** $30°$
**b.** $45°$ Correct Choice
**c.** $60°$
**d.** $120°$
**e.** $135°$

**Solution**: $\overrightarrow{AB} = B - A = \left(1, 1, \sqrt{2}\right)$ $\overrightarrow{AC} = C - A = (2, 2, 0)$.

$\overrightarrow{AB} \cdot \overrightarrow{AC} = 2 + 2 = 4$ $\left|\overrightarrow{AB}\right| = \sqrt{1 + 1 + 2} = 2$ $\left|\overrightarrow{AC}\right| = \sqrt{4 + 4} = 2\sqrt{2}$

$$\cos\alpha = \frac{\overrightarrow{AB} \cdot \overrightarrow{AC}}{\left|\overrightarrow{AB}\right|\left|\overrightarrow{AC}\right|} = \frac{4}{2 \cdot 2\sqrt{2}} = \frac{1}{\sqrt{2}} \qquad \alpha = \arccos\left(\frac{1}{\sqrt{2}}\right) = 45°$$

**2.** The vertices of a triangle are $A = \left(2, 1, \sqrt{2}\right)$, $B = \left(3, 2, 2\sqrt{2}\right)$ and $C = \left(4, 3, \sqrt{2}\right)$. Find a vector perpendicular to the plane of this triangle.

**a.** $(1, -1, 0)$ Correct Choice
**b.** $(1, 1, 0)$
**c.** $(1, -1, 1)$
**d.** $(1, 1, 1)$
**e.** $(-1, -1, 1)$

**Solution**: $\overrightarrow{AB} = \left(1, 1, \sqrt{2}\right)$ $\overrightarrow{AC} = (2, 2, 0)$.

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 1 & \sqrt{2} \\ 2 & 2 & 0 \end{vmatrix} = \hat{\imath}\left(0 - 2\sqrt{2}\right) - \hat{\jmath}\left(0 - 2\sqrt{2}\right) + \hat{k}(0)$$

$$= \left(-2\sqrt{2}, 2\sqrt{2}, 0\right) = -2\sqrt{2}\,(1, -1, 0)$$

**3**. Which of the following points lies on the line $(x,y,z) = (2-t, 3+2t, 4+t)$ and on the plane $2x+3y+4z = 21$?

- **a**. $(1,1,1)$
- **b**. $(4,3,2)$
- **c**. $(2,3,2)$
- **d**. $(3,1,3)$ Correct Choice
- **e**. $(2,2,2)$

**Solution**: We are looking for the intersection. Plug the line into the plane:
$2(2-t)+3(3+2t)+4(4+t) = 21$ So $29+8t = 21$ or $8t = -8$ or $t = -1$.
So the point of intersection is $(x,y,z) = (2+1, 3-2, 4-1) = (3,1,3)$

**4**. The quadratic surface $x^2 - y^2 - 6x + 4y + 2 = 0$ is a

- **a**. hyperboloid
- **b**. hyperbolic ellipsoid
- **c**. hyperbola
- **d**. hyperboic paraboloid
- **e**. hyperbolic cylinder Correct Choice

**Solution**: Since there are no $z$'s, this surface is a cylinder. Since the equation is a hyperbola in the $xy$-plane, this surface is a hyperbolic cylinder.

**5**. For the "twisted cubic" curve $\vec{r}(t) = \left(t, t^2, \frac{2}{3}t^3\right)$, find the binormal vector $\hat{B}$.

- **a**. $\left(\frac{2t^2}{2t^2+1}, \frac{-2t}{2t^2+1}, \frac{1}{2t^2+1}\right)$ Correct Choice
- **b**. $\left(\frac{1}{2t^2+1}, \frac{2t}{2t^2+1}, \frac{2t^2}{2t^2+1}\right)$
- **c**. $\left(\frac{-2t^2}{2t^2+1}, \frac{2t}{2t^2+1}, \frac{-1}{2t^2+1}\right)$
- **d**. $\left(\frac{1}{2t^2+1}, \frac{-2t}{2t^2+1}, \frac{2t^2}{2t^2+1}\right)$
- **e**. $\left(\frac{2t^2}{2t^2+1}, \frac{2t}{2t^2+1}, \frac{1}{2t^2+1}\right)$

**Solution**: $\vec{v} = (1, 2t, 2t^2)$ $\vec{a} = (0, 2, 4t)$

$$\vec{v}\times\vec{a} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 2t & 2t^2 \\ 0 & 2 & 4t \end{vmatrix} = \hat{\imath}(8t^2-4t^2) - \hat{\jmath}(4t-0) + \hat{k}(2-0) = (4t^2, -4t, 2)$$

$$|\vec{v}\times\vec{a}| = \sqrt{16t^4+16t^2+4} = 4t^2+2 \qquad \hat{B} = \frac{\vec{v}\times\vec{a}}{|\vec{v}\times\vec{a}|} = \left(\frac{2t^2}{2t^2+1}, \frac{-2t}{2t^2+1}, \frac{1}{2t^2+1}\right)$$

**6**. Find the mass of the "twisted cubic" curve $\vec{r}(t) = \left(t, t^2, \frac{2}{3}t^3\right)$ between $t = 0$ and $t = 1$ if the linear density is $\rho = y^2 + 6xz$.

**a**. $1$
**b**. $\frac{1}{5}$
**c**. $\frac{7}{5}$
**d**. $\frac{20}{7}$
**e**. $\frac{17}{7}$ Correct Choice

**Solution**: $\rho = y^2 + 6xz \qquad \rho(\vec{r}(t)) = (t^2)^2 + 6(t)\left(\frac{2}{3}t^3\right) = 5t^4$

$\vec{v} = (1, 2t, 2t^2) \qquad |\vec{v}| = \sqrt{1 + 4t^2 + 4t^4} = 1 + 2t^2$

$M = \int \rho\, ds = \int \rho(\vec{r}(t))\,|\vec{v}|\,dt = \int_0^1 5t^4(1 + 2t^2)\,dt = \int_0^1 5t^4 + 10t^6\,dt$

$= \left[t^5 + 10\frac{t^7}{7}\right]_0^1 = 1 + \frac{10}{7} = \frac{17}{7}$

**7**. Find the work done when a bead is pushed along the "twisted cubic" curve $\vec{r}(t) = \left(t, t^2, \frac{2}{3}t^3\right)$ between $t = 0$ and $t = 1$ if you apply the force $\vec{F} = (3z, y, x)$.

**a**. $\frac{1}{2}$
**b**. $1$
**c**. $\frac{3}{2}$ Correct Choice
**d**. $2$
**e**. $\frac{5}{2}$

**Solution**: $\vec{F} = (3z, y, x) \qquad \vec{F}(\vec{r}(t)) = (2t^3, t^2, t) \qquad \vec{v} = (1, 2t, 2t^2)$

$W = \int \vec{F} \cdot d\vec{s} = \int \vec{F}(\vec{r}(t)) \cdot \vec{v}\,dt = \int_0^1 (2t^3 + 2t^3 + 2t^3)\,dt = \int_0^1 6t^3\,dt = \left[\frac{6t^4}{4}\right]_0^1 = \frac{3}{2}$

**8**. You are riding on a train which is currently travelling EAST but curving toward the SOUTH. Where do $\hat{B}$ and $\hat{N}$ for the train currently point?

**a**. $\hat{B}$ points SOUTH and $\hat{N}$ points DOWN.
**b**. $\hat{B}$ points SOUTH and $\hat{N}$ points UP.
**c**. $\hat{B}$ points UP and $\hat{N}$ points SOUTH.
**d**. $\hat{B}$ points DOWN and $\hat{N}$ points SOUTH. Correct Choice
**e**. $\hat{B}$ points DOWN and $\hat{N}$ points SOUTHEAST.

**Solution**: $\vec{v}$ and $\hat{T}$ point EAST and $\vec{a}$ points horizontally to the right of East. So $\hat{N}$ points SOUTH and $\hat{B} = \hat{T} \times \hat{N}$ points DOWN.

**9**. For the function $f = x\sin(yz)$, which of the following are correct?

I. $\dfrac{\partial^2 f}{\partial x \partial y} = -z\cos yz$ III. $\dfrac{\partial^2 f}{\partial x \partial z} = y\cos yz$ V. $\dfrac{\partial^2 f}{\partial y \partial z} = x\cos yz - xyz\sin yz$

II. $\dfrac{\partial^2 f}{\partial y \partial x} = z\cos yz$ IV. $\dfrac{\partial^2 f}{\partial z \partial x} = y\cos yz$ VI. $\dfrac{\partial^2 f}{\partial z \partial y} = x\cos yz + xyz\sin yz$

**a**. I and II.
**b**. III and IV. Correct Choice
**c**. V and VI.
**d**. I, II and III.
**e**. IV, V and VI.

**Solution**: Since mixed partial derivatives are equal, I and II cannot both be correct and V and VI cannot both be correct.

**10**. Find the equation of the plane tangent to the graph of the function $z = f(x,y) = x^2y + xy^3$ at $(x,y) = (2,1)$. What is the $z$-intercept?

**a**. $-14$ Correct Choice
**b**. $-6$
**c**. $6$
**d**. $14$
**e**. $26$

**Solution**: $f(x,y) = x^2y + xy^3$ $f(2,1) = 6$ $z = f_{\text{tan}}(x,y)$

$f_x(x,y) = 2xy + y^3$ $f_x(2,1) = 5$ $z = f(2,1) + f_x(2,1)(x-2) + f_y(2,1)(y-1)$

$f_y(x,y) = x^2 + 3xy^2$ $f_y(2,1) = 10$ $z = 6 + 5(x-2) + 10(y-1)$

$z$-intercept is $c = 6 + 5(-2) + 10(-1) = -14$

**11**. Find the equation of the plane tangent to the graph of the equation $x\sin(yz) = 1$ at $P = \left(\sqrt{2}, \frac{1}{4}, \pi\right)$. What is the $z$-intercept?

**a**. $\sqrt{2} + \frac{\pi}{4}$
**b**. $1 + \frac{\pi}{2}$
**c**. $2 + \pi$
**d**. $4 + 2\pi$ Correct Choice
**e**. $2\sqrt{2} + 2\pi$

**Solution**: The graph is a level set of the function $F = x\sin(yz)$.

$\vec{\nabla} F = (\sin(yz), xz\cos(yz), xy\cos(yz))$

$\vec{N} = \vec{\nabla} F\big|_P = \left(\sin\left(\frac{\pi}{4}\right), \sqrt{2}\,\pi\cos\left(\frac{\pi}{4}\right), \sqrt{2}\,\frac{1}{4}\cos\left(\frac{\pi}{4}\right)\right) = \left(\frac{1}{\sqrt{2}}, \pi, \frac{1}{4}\right)$

$\vec{N}\cdot X = \vec{N}\cdot P$ $\frac{1}{\sqrt{2}}x + \pi y + \frac{1}{4}z = \frac{1}{\sqrt{2}}\cdot\sqrt{2} + \pi\cdot\frac{1}{4} + \frac{1}{4}\cdot\pi = 1 + \frac{\pi}{2}$

$z$-intercept is $c = 4\left(1 + \frac{\pi}{2}\right) = 4 + 2\pi$

**12**. A fish is currently at the point $(x,y,z) = (1,2,-3)$ and has velocity $\vec{v} = (1,2,1)$. If the salt density is $D = xyz^2$, find $\frac{dD}{dt}$, the time rate of change of the density as seen by the fish at the current instant.

**a**. 12
**b**. 24 Correct Choice
**c**. 36
**d**. 48
**e**. 60

**Solution**: $\vec{\nabla}D = (yz^2, xz^2, 2xyz) = (18, 9, -12)$

$\frac{dD}{dt} = \vec{v} \cdot \vec{\nabla}D = (1,2,1) \cdot (18,9,-12) = 18 + 18 - 12 = 24$

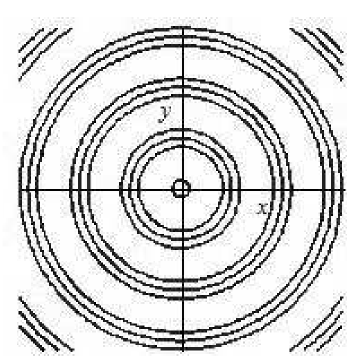
**13**. The equation $z^3 \sin x + z\cos y = 3$ defines $z$ as an implicit function of $x$ and $y$. Notice that its graph passes through the point $\left(\frac{\pi}{4}, \frac{\pi}{4}, \sqrt{2}\right)$. Find $\frac{\partial z}{\partial y}$ at $\left(\frac{\pi}{4}, \frac{\pi}{4}\right)$.

**a**. $\frac{\sqrt{2}}{5}$
**b**. $\frac{\sqrt{2}}{6}$
**c**. $\frac{\sqrt{2}}{7}$ Correct Choice
**d**. $\frac{1}{6}$
**e**. $\frac{1}{7}$

**Solution**: Apply $\frac{\partial}{\partial y}$ to both sides: $3z^2 \frac{\partial z}{\partial y} \sin x + \frac{\partial z}{\partial y} \cos y - z\sin y = 0$

$6\frac{\partial z}{\partial y}\frac{1}{\sqrt{2}} + \frac{\partial z}{\partial y}\frac{1}{\sqrt{2}} - \sqrt{2}\frac{1}{\sqrt{2}} = 0 \qquad 7\frac{\partial z}{\partial y} - \sqrt{2} = 0 \qquad \frac{\partial z}{\partial y} = \frac{\sqrt{2}}{7}$

**14**. The plot at the right is the contour plot of which of these functions?




**a**. $f(x,y) = \sin(x)\sin(y)$
**b**. $f(x,y) = x^2 - y^2$
**c**. $f(x,y) = \sin\left(\sqrt{x^2+y^2}\right)$ Correct Choice
**d**. $f(x,y) = \sin(x) + \sin(y)$
**e**. $f(x,y) = \sin(xy)$

**Solution**: Since the plot is circularly symmetric, the function must be a function of only the polar coordinate $r = \sqrt{x^2+y^2}$.

Work Out: (10 points each. Part credit possible. Show all work.)

**15**. The pressure $P$, the temperature $T$, and the density $\rho$, of a certain ideal gas are related by $P = 10^{-3}\rho T$. Currently, the temperature is $T = 300°\text{K}$ and is increasing at $2°\text{K}$ per minute while the density is $\rho = 4\dfrac{\text{gm}}{\text{cm}^3}$ and is decreasing at $0.05\dfrac{\text{gm}}{\text{cm}^3}$ per minute. Consequently, the pressure is currently $P = 10^{-3}\rho T = 10^{-3}(4)(300) = 1.2$ atm. At what rate is $P$ changing and is it increasing or decreasing?

**Solution**: $\dfrac{dT}{dt} = 2 \qquad \dfrac{d\rho}{dt} = -0.05 \qquad \dfrac{\partial P}{\partial T} = 10^{-3}\rho = .004 \qquad \dfrac{\partial P}{\partial \rho} = 10^{-3}T = 0.3$

By chain rule: $\dfrac{dP}{dt} = \dfrac{\partial P}{\partial T}\dfrac{dT}{dt} + \dfrac{\partial P}{\partial \rho}\dfrac{d\rho}{dt} = .004 \times 2 + 0.3 \times (-0.05) = .008 - .015 = -0.007$

So $P$ is decreasing at $0.007$ atm per minute.

**16**. The volume of a cone is $V = \dfrac{1}{3}\pi r^2 h$. If the radius and height are measured to be $r = 3\text{cm} \pm 0.02\text{cm}$ and $h = 5\text{cm} \pm 0.03\text{cm}$, then the volume is computed to be $V = \dfrac{1}{3}\pi 3^2 5 = 15\pi\text{cm}^3$. Use differentials to estimate the error in this computed volume.

**Solution**: $\Delta V \approx dV = \dfrac{\partial V}{\partial r}dr + \dfrac{\partial V}{\partial h}dh = \dfrac{2}{3}\pi r h\, dr + \dfrac{1}{3}\pi r^2\, dh$

$= \dfrac{2}{3}\pi(3)(5)(.02) + \dfrac{1}{3}\pi(3)^2(.03) = 0.29\pi = 0.91106$

**17**. Find the minimum value of the function $f = x^2 + 2y^2 + 4z^2$ on the plane $x + y + z = 14$.

**Solution**:

METHOD I: Eliminate a Variable:

$x = 14 - y - z \qquad f = (14 - y - z)^2 + 2y^2 + 4z^2$

$\dfrac{\partial f}{\partial y} = -2(14 - y - z) + 4y = 0 \qquad \dfrac{\partial f}{\partial z} = -2(14 - y - z) + 8z = 0$

$6y + 2z = 28 \qquad 2y + 10z = 28 \qquad \Rightarrow \qquad y = 4, z = 2$

So $x = 8$ and $f(8,4,2) = 64 + 32 + 16 = 112$

There is only one critical point and $f$ can be arbitrarily large.

So the critical point must be the minimum.

METHOD II: Lagrange Multipliers

$\vec{\nabla} f = (2x, 4y, 8z) \qquad \vec{\nabla} g = (1,1,1) \qquad \vec{\nabla} f = \lambda \vec{\nabla} g \qquad 2x = \lambda \qquad 4y = \lambda \qquad 8z = \lambda$

$\lambda = 2x = 4y = 8z \qquad x = 4z \qquad y = 2z$

Use the constraint: $4z + 2z + z = 14 \qquad 7z = 14 \qquad z = 2 \qquad x = 8 \qquad y = 4$

$f(8,4,2) = 64 + 32 + 16 = 112$

There is only one critical point and $f$ can be arbitrarily large.

So the critical point must be the minimum.