Name________________________

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10	/20
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Total	/103

MATH 253H Final Exam Fall 2014

Sections 201-202 Solutions P. Yasskin

Multiple Choice: (4 points each. No part credit.)

1. If $z = f(x,y)$ where $x = r\cos\theta$ and $y = r\sin\theta$ then

a. $\dfrac{\partial z}{\partial r} = -\dfrac{\partial f}{\partial x}\sin\theta + \dfrac{\partial f}{\partial y}\cos\theta$

b. $\dfrac{\partial z}{\partial r} = \dfrac{\partial f}{\partial x}r\cos\theta + \dfrac{\partial f}{\partial y}r\sin\theta$

c. $\dfrac{\partial z}{\partial \theta} = \dfrac{\partial f}{\partial x}r\sin\theta - \dfrac{\partial f}{\partial y}r\cos\theta$

d. $\dfrac{\partial z}{\partial \theta} = -\dfrac{\partial f}{\partial x}r\sin\theta + \dfrac{\partial f}{\partial y}r\cos\theta$ Correct Choice

e. $\dfrac{\partial z}{\partial \theta} = \dfrac{\partial f}{\partial x}r\cos\theta + \dfrac{\partial f}{\partial y}r\sin\theta$

Solution: $\dfrac{\partial z}{\partial r} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial r} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial r} = \dfrac{\partial f}{\partial x}\cos\theta + \dfrac{\partial f}{\partial y}\sin\theta$

$\dfrac{\partial z}{\partial \theta} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial \theta} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial \theta} = -\dfrac{\partial f}{\partial x}r\sin\theta + \dfrac{\partial f}{\partial y}r\cos\theta$

2. If a car is currently at the point $P = (3,-4)$ and has velocity $\vec{v} = (-2,1)$ at what rate is its distance from the origin currently changing?

a. $\dfrac{dD}{dt} = -2$ Correct Choice

b. $\dfrac{dD}{dt} = -\dfrac{2}{5}$

c. $\dfrac{dD}{dt} = \dfrac{2}{5}$

d. $\dfrac{dD}{dt} = 2$

e. $\dfrac{dD}{dt} = \dfrac{11}{5}$

Solution: $D = \sqrt{x^2+y^2}$ $\quad \dfrac{\partial D}{\partial x} = \dfrac{x}{\sqrt{x^2+y^2}} = \dfrac{3}{5}$ $\quad \dfrac{\partial D}{\partial y} = \dfrac{y}{\sqrt{x^2+y^2}} = \dfrac{-4}{5}$

$\dfrac{dD}{dt} = \dfrac{\partial D}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial D}{\partial y}\dfrac{dy}{dt} = \dfrac{3}{5}\cdot(-2) - \dfrac{4}{5}(1) = \dfrac{-6-4}{5} = -2$

3. Find the point where the line $\vec{r}(t) = (4t+1, 2t-1, 3t)$ intersects the plane $\vec{R}(u,v) = (3+u, 2-v, u+v)$. At this point (x,y,z) what is $x+y+z$?

- **a.** -9
- **b.** 0
- **c.** 9 Correct Choice
- **d.** 18
- **e.** 45

Solution: $\vec{r}(t) = \vec{R}(u,v)$: $4t+1 = 3+u$ $2t-1 = 2-v$ $3t = u+v$
So $u = 4t-2$ $v = 3-2t$ and $3t = (4t-2)+(3-2t) = 2t+1$ $t = 1$
$(x,y,z) = \vec{r}(1) = (5,1,3)$ $x+y+z = 9$

4. Ham Duet is flying the Centenial Eagle above the Death Star along the curve $\vec{r}(t) = (t^2, t^3, 8-t^2)$. At $t = 2$, Ham releases a space torpedo which then glides with constant velocity equal to the velocity of the Centenial Eagle at the instant of release. Where does the torpedo hit the surface of the Death Star, which is at $z = 0$?

- **a.** $(12, 32, -4)$
- **b.** $(8, 20, 0)$ Correct Choice
- **c.** $(4, 8, 4)$
- **d.** $(0, -4, 8)$
- **e.** $\left(8, 8+6\sqrt{2}, 0\right)$

Solution: $\vec{r}(t) = (t^2, t^3, 8-t^2)$, $\vec{r}(2) = (4,8,4)$, $\vec{v}(t) = (2t, 3t^2, -2t)$ $\vec{v}(2) = (4,12,-4)$
$X = P + t\vec{v} = (4,8,4) + t(4,12,-4) = (4+4t, 8+12t, 4-4t)$
It hits the surface when $z = 4-4t = 0$. Or $t = 1$. So $X = (8,20,0)$.

5. Duke Skywater is flying the Centenial Eagle through a dangerous polaron field whose density is given by $\delta = \dfrac{1}{\sqrt{x^2+y^2+z^2}}$. If Duke is currently at the point $P = (1,-2,2)$ in what unit vector direction should he fly to **REDUCE** the polaron density as fast as possible?

- **a.** $\left(\frac{2}{3}, \frac{-1}{3}, \frac{1}{3}\right)$
- **b.** $\left(\frac{-2}{3}, \frac{1}{3}, \frac{-1}{3}\right)$
- **c.** $\left(\frac{-1}{3}, \frac{-2}{3}, \frac{-2}{3}\right)$
- **d.** $\left(\frac{-1}{3}, \frac{2}{3}, \frac{-2}{3}\right)$
- **e.** $\left(\frac{1}{3}, \frac{-2}{3}, \frac{2}{3}\right)$ Correct Choice

Solution: $\vec{\nabla}\rho = \left(\dfrac{-x}{(x^2+y^2+z^2)^{3/2}}, \dfrac{-y}{(x^2+y^2+z^2)^{3/2}}, \dfrac{-z}{(x^2+y^2+z^2)^{3/2}}\right)$ $\vec{\nabla}\rho\Big|_{(1,-2,2)} = \left(\frac{-1}{27}, \frac{2}{27}, \frac{-2}{27}\right)$
$\left|\vec{\nabla}\rho\right| = \frac{1}{27}\sqrt{1+4+4} = \frac{1}{9}$
Fastest DECREASE along: $\vec{u} = -\dfrac{1}{\left|\vec{\nabla}\rho\right|}\vec{\nabla}\rho\Big|_{(1,2,3)} = -9\left(\frac{-1}{27}, \frac{2}{27}, \frac{-2}{27}\right) = \left(\frac{1}{3}, \frac{-2}{3}, \frac{2}{3}\right)$

6. Queen Leah is flying the Centenial Eagle above Planet Tattoo along the curve $\vec{r}(t) = (t^2, t^3, 8 - t^2)$. What is her tangential acceleration, a_T, at $t = \frac{1}{3}$?

a. $\frac{10}{3}$ Correct Choice
b. $\frac{5}{3}$
c. $\frac{10}{9}$
d. $\sqrt{3}$
e. $2\sqrt{3}$

Solution: $\vec{r}(t) = (t^2, t^3, 8 - t^2)$ $\vec{v}(t) = (2t, 3t^2, -2t)$ $|\vec{v}| = \sqrt{4t^2 + 9t^4 + 4t^2} = \sqrt{8t^2 + 9t^4}$

$$a_T = \frac{d}{dt}|\vec{v}| = \frac{16t + 36t^3}{2\sqrt{8t^2 + 9t^4}} \qquad a_T\left(\frac{1}{3}\right) = \frac{\frac{16}{3} + \frac{36}{27}}{2\sqrt{\frac{8}{9} + \frac{1}{9}}} = \frac{1}{2}\left(\frac{16}{3} + \frac{4}{3}\right) = \frac{10}{3}$$

7. Find the tangent plane to the graph of $z = \frac{1}{xy}$ at $(2,3)$. The z-intercept is

a. 0
b. $\frac{1}{2}$ Correct Choice
c. 1
d. $\frac{1}{6}$
e. $\frac{1}{36}$

Solution: $f(x,y) = \frac{1}{xy}$ $f_x(x,y) = \frac{-1}{x^2y}$ $f_y(x,y) = \frac{-1}{xy^2}$

$f(2,3) = \frac{1}{6}$ $f_x(2,3) = \frac{-1}{12}$ $f_y(2,3) = \frac{-1}{18}$

$$z = f_{\tan}(x,y) = f(2,3) + f_x(2,3)(x-2) + f_y(2,3)(y-3)$$

$$z = \frac{1}{6} - \frac{1}{12}(x-2) - \frac{1}{18}(y-3) \qquad c = \frac{1}{6} - \frac{1}{12}(-2) - \frac{1}{18}(-3) = \frac{1}{2}$$

8. Find the tangent plane to the graph of the ellipsoid $x^2 + 3xy + 2y^2 + z^2 = 21$ at $(2,1,3)$. The z-intercept is

a. 3
b. 4
c. 6
d. 7 Correct Choice
e. 42

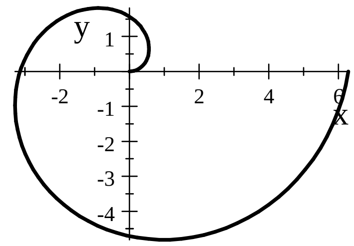
Solution: $F = x^2 + 3xy + 2y^2 + z^2$ $P = (2,1,3)$ $\vec{\nabla}F = (2x + 3y, 3x + 4y, 2z)$

$\vec{N} = \vec{\nabla}F\Big|_P = (7, 10, 6)$ $\vec{N} \cdot X = \vec{N} \cdot P$ $7x + 10y + 6z = 7(2) + 10(1) + 6(3) = 42$

$6c = 42$ $c = 7$

9. A wire has the shape of the spiral given in polar coordinates by $r = \theta$ from $\underline{\theta = 0 \text{ to } \theta = \pi}$ and has linear density $\rho = \sqrt{x^2 + y^2}$. Find its mass.

Hint: Parametrize the curve.



a. $\frac{1}{3}(1+\pi^2)^{3/2} - \frac{1}{3}$ Correct Choice

b. $\frac{2}{3}(1+\pi^2)^{3/2} - \frac{2}{3}$

c. $\frac{4}{3}(1+\pi^2)^{3/2} - \frac{4}{3}$

d. $\frac{2}{3}(1+\pi^2)^{3/2}$

e. $\frac{4}{3}(1+\pi^2)^{3/2}$

Solution: $\vec{r}(\theta) = (\theta\cos\theta, \theta\sin\theta)$ $\quad \vec{v} = (\cos\theta - \theta\sin\theta, \sin\theta + \theta\cos\theta)$

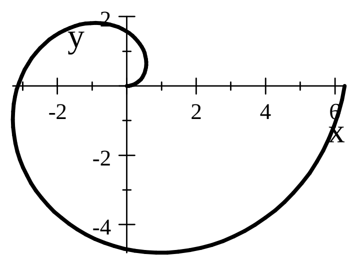
$|\vec{v}| = \sqrt{(\cos\theta - \theta\sin\theta)^2 + (\sin\theta + \theta\cos\theta)^2} = \sqrt{\cos^2\theta + \theta^2\sin^2\theta + \sin^2\theta + \theta^2\cos^2\theta} = \sqrt{1+\theta^2}$

$\rho = r = \theta \qquad M = \int \rho\, ds = \int_0^\pi \rho|\vec{v}|\, d\theta = \int_0^\pi \theta\sqrt{1+\theta^2}\, d\theta$

$= \left[\frac{1}{3}(1+\theta^2)^{3/2}\right]_0^\pi = \frac{1}{3}(1+\pi^2)^{3/2} - \frac{1}{3}$

10. Compute $\int \vec{F} \cdot d\vec{s}$ for $\vec{F} = (2x, 2y)$ counterclockwise around the piece of the spiral given in polar coordinates by $r = \theta$ from $\underline{\theta = \pi \text{ to } \theta = 2\pi}$.

Hint: Use a theorem.



a. π^2

b. $2\pi^2$

c. $3\pi^2$ Correct Choice

d. $4\pi^2$

e. $5\pi^2$

Solution: To use the Fundamental Theorem of Calculus for Curves, we need a scalar potential and the endpoints.

$\vec{F} = \vec{\nabla} f \qquad \partial_x f = 2x \qquad \partial_y f = 2y \qquad f = x^2 + y^2$

$r = \theta = \pi$ is the point $(x,y) = (r\cos\theta, r\sin\theta) = (-\pi, 0)$

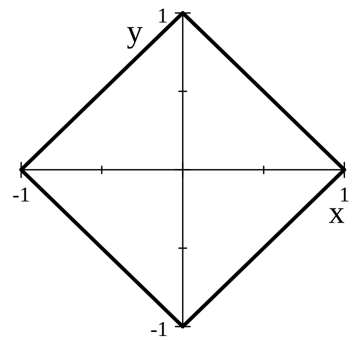
$r = \theta = 2\pi$ is the point $(x,y) = (r\cos\theta, r\sin\theta) = (2\pi, 0)$

$\int \vec{F} \cdot d\vec{s} = \int \vec{\nabla} f \cdot d\vec{s} = f(2\pi, 0) - f(-\pi, 0) = (2\pi)^2 - (-\pi)^2 = 3\pi^2$

11. Compute $\iint_D e^{x-y}\,dA$ over the region D of all points (x,y) such that $|x|+|y|\le 1$.
Hints: Draw the region and use a change of coordinates with $u = x+y$ and $v = x-y$.

a. $2e+\dfrac{2}{e}$
b. $e+\dfrac{1}{e}$
c. $2e-\dfrac{2}{e}$
d. $e-\dfrac{1}{e}$ Correct Choice
e. $\dfrac{1}{e}-e$

Solution: Plot in each quadrant:
I: $x+y=1$ II: $-x+y=1$
III: $-x-y=1$ IV: $x-y=1$
or
I: $u=1$ II: $v=-1$
III: $u=-1$ IV: $v=1$



$u+v=2x \qquad u-v=2y \qquad x=\dfrac{u+v}{2} \qquad y=\dfrac{u-v}{2}$

$$J=\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left\|\begin{matrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & -\frac{1}{2}\end{matrix}\right\| = \left|-\frac{1}{4}-\frac{1}{4}\right| = \frac{1}{2}$$

$$\iint_D e^{x-y}\,dA = \int_{-1}^{1}\int_{-1}^{1} e^{v}\,\frac{1}{2}\,du\,dv = \left[\frac{u}{2}\right]_{-1}^{1}\left[e^{v}\right]_{-1}^{1} = e-\frac{1}{e}$$

12. Compute the integral $\iint \vec{F}\cdot d\vec{S}$ for $\vec{F}=(2xz^2, yz^2, z^3)$ over the complete surface of the solid between the paraboloid $z=x^2+y^2$ and the plane $z=4$, with outward normal.
Hint: Use a theorem.

a. 96π
b. 192π
c. 384π Correct Choice
d. 768π
e. 0

Solution: By Gauss' Theorem, $\iint_{\partial V}\vec{F}\cdot d\vec{S} = \iiint_V \vec{\nabla}\cdot\vec{F}\,dV$
We do the volume integral in cylindrical coordinates.
The bottom and top surfaces are $z=r^2$ and $z=4$ which intersect when $r^2=4 \qquad r=2$

$\vec{\nabla}\cdot\vec{F} = 2z^2+z^2+3z^2 = 6z^2 \qquad dV = r\,dr\,d\theta\,dz$

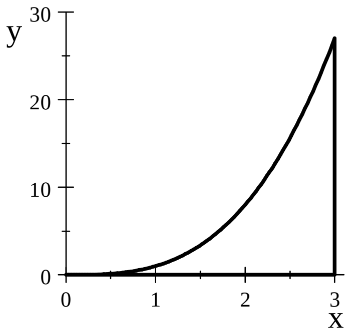
$$\iiint_V \vec{\nabla}\cdot\vec{F}\,dV = \int_0^{2\pi}\int_0^{2}\int_{r^2}^{4} 6z^2 r\,dz\,dr\,d\theta = 2\pi\int_0^2\left[2z^3\right]_{z=r^2}^{4} r\,dr = 4\pi\int_0^2\left[4^3-(r^2)^3\right]r\,dr$$
$$= 4\pi\int_0^2 64r - r^7\,dr = 4\pi\left[32r^2-\frac{r^8}{8}\right]_0^2 = 4\pi(128-32) = 384\pi$$

Work Out: (Points indicated. Part credit possible. Show all work.)

13. (20 points) For each integral, plot the region of integration and then compute the integral.

a. $I = \int_0^{27} \int_{\sqrt[3]{y}}^{3} e^{-x^4}\,dx\,dy$

Solution:



$$I = \int_0^3 \int_0^{x^3} e^{-x^4}\,dy\,dx = \int_0^3 \left[e^{-x^4} y \right]_{y=0}^{x^3} dx$$

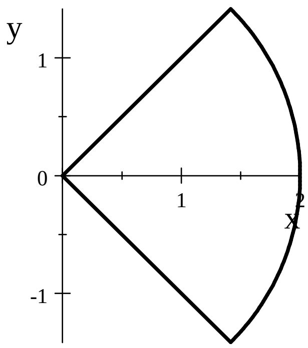
$$= \int_0^3 e^{-x^4} x^3\,dx$$

$$u = x^4 \qquad du = 4x^3\,dx \qquad x^3\,dx = \frac{1}{4}\,du$$

$$I = \frac{1}{4} \int_0^{81} e^{-u} du = \frac{1}{4} [-e^{-u}]_0^{81} = \frac{1}{4}(1 - e^{-81})$$

b. $J = \int_{-\sqrt{2}}^{\sqrt{2}} \int_{|y|}^{\sqrt{4-y^2}} e^{-x^2-y^2}\,dx\,dy$

Solution:



$$J = \int_{-\pi/4}^{\pi/4} \int_0^2 e^{-r^2}\, r\,dr\,d\theta$$

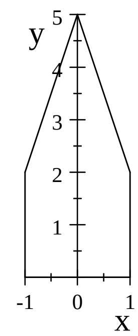
$$u = r^2 \qquad du = 2r\,dr \qquad r\,dr = \frac{1}{2}\,du$$

$$J = \frac{\pi}{2} \frac{1}{2} \int_0^4 e^{-u}\,du$$

$$= \frac{\pi}{4} [-e^{-u}]_0^4 = \frac{\pi}{4}(1 - e^{-4})$$

14. (10 points) Compute $\oint \vec{F} \cdot d\vec{s}$ for $\vec{F} = (3y - 2x^2, 3y^2 - 2x)$ over the complete boundary of the shape at the right, which is a square of side 2 under an isosceles triangle with height 3.

If you use a theorem, name it.



Solution: Green's Theorem says $\oint_{\partial R} P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy$

Here R is the region inside the square and triangle and

$\oint \vec{F} \cdot d\vec{s} = \oint_{\partial R} (3y - 2x^2)\,dx + (3y^2 - 2x)\,dy$ So $P = 3y - 2x^2$ and $Q = 3y^2 - 2x$.

$$\oint \vec{F} \cdot d\vec{s} = \iint_R \left(\frac{\partial}{\partial x}(3y^2 - 2x) - \frac{\partial}{\partial y}(3y - 2x^2) \right) dx\,dy = \iint_R ((-2) - (3))\,dx\,dy$$

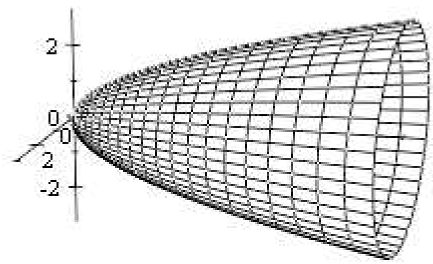
$$= -5 \iint_R 1\,dx\,dy = -5(Area) = -5\left(2^2 + \frac{1}{2} 2 \cdot 3\right) = -35$$

15. (25 points) Verify Stokes' Theorem $\iint_P \vec{\nabla} \times \vec{F} \bullet d\vec{S} = \oint_{\partial P} \vec{F} \bullet d\vec{s}$

for the vector field $\vec{F} = (x^2 z, x - z, -z^2 x)$ and the paraboloid

$y = x^2 + z^2$ with $y \le 9$ oriented to the right

Use the following steps:



First the Left Hand Side:

Parametrize the paraboloid as:

$$\vec{R}(r,\theta) = (r\cos\theta, r^2, r\sin\theta)$$

a. Compute the coordinate tangent vectors:

$$\begin{array}{llll} & \hat{\imath} & \hat{\jmath} & \hat{k} \\ \vec{e}_r = (& \cos\theta, & 2r, & \sin\theta) \\ \vec{e}_\theta = (& -r\sin\theta, & 0, & r\cos\theta) \end{array}$$

b. Compute the normal vector:

$$\vec{N} = \hat{\imath}(2r^2\cos\theta - 0) - \hat{\jmath}(r\cos^2\theta - -r\sin^2\theta) + \hat{k}(0 - -2r^2\sin\theta) = (2r^2\cos\theta, -r, 2r^2\sin\theta)$$

Reverse $\vec{N} = (-2r^2\cos\theta, r, -2r^2\sin\theta)$

c. Compute the curl:

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ x^2 z & x - z & -z^2 x \end{vmatrix} = \hat{\imath}(0 - -1) - \hat{\jmath}(-z^2 - x^2) + \hat{k}(1 - 0) = (1, z^2 + x^2, 1)$$

d. Evaluate the curl on the surface:

$$\vec{\nabla} \times \vec{F}\Big|_{\vec{R}(r,\theta)} = (1, r^2, 1)$$

e. Compute the left hand side:

$$LHS = \iint_P \vec{\nabla} \times \vec{F} \bullet d\vec{S} = \int_0^{2\pi}\int_0^3 \vec{\nabla} \times \vec{F} \bullet \vec{N}\, dr\, d\theta = \int_0^{2\pi}\int_0^3 -2r^2\cos\theta + r^3 - 2r^2\sin\theta\, dr\, d\theta$$

Do the θ integral first:

$$LHS = \int_0^3 \left[-2r^2\sin\theta + r^3\theta + 2r^2\cos\theta\right]_{\theta=0}^{2\pi} dr = 2\pi\int_0^3 r^3\, dr = \left[\frac{\pi}{2}r^4\right]_0^3 = \frac{81\pi}{2}$$

Second the Right Hand Side:

f. Parametrize the boundary circle:

$$\vec{r}(\theta) = (3\cos\theta, 9, 3\sin\theta)$$

g. Compute the tangent vector:

$\vec{v} = (-3\sin\theta, 0, 3\cos\theta)$ We need to orient counterclockwise as seen from the positive y axis.

$\vec{r}(0) = (3,9,0)$ $\quad \vec{r}\left(\frac{\pi}{2}\right) = (0,9,3)$ which goes from $+x$ axis to $+z$ axis, or clockwise.

Reverse $\vec{v} = (3\sin\theta, 0, -3\cos\theta)$

h. Evaluate $\vec{F} = (x^2 z, x - z, -z^2 x)$ on the circle:

$$\vec{F}\Big|_{\vec{r}(\theta)} = (27\cos^2\theta\sin\theta, 3\cos\theta - 3\sin\theta, -27\sin^2\theta\cos\theta)$$

i. Compute the right hand side:

$$RHS = \oint_{\partial P} \vec{F} \cdot d\vec{s} = \int_0^{2\pi} \vec{F} \cdot \vec{v}\, d\theta = \int_0^{2\pi} (81\sin^2\theta\cos^2\theta + 81\sin^2\theta\cos^2\theta)\, d\theta = \int_0^{2\pi} 162\sin^2\theta\cos^2\theta\, d\theta$$

$$= \frac{81}{2}\int_0^{2\pi} \sin^2(2\theta)\, d\theta = \frac{81}{2}\,\frac{1}{2}\,2\pi = \frac{81}{2}\pi$$

which agrees with the left hand side.