1-12	/60
13	/20
14	/20
Total	/100

Name________________________

MATH 253 Exam 1 Fall 2016

Sections 201/202 Solutions P. Yasskin

Multiple Choice: (5 points each. No part credit.)

1. Find the distance from the point $\langle 3,4,12 \rangle$ to the sphere $x^2 + y^2 + z^2 = 64$.

- **a**. 1
- **b**. 5 Correct
- **c**. 8
- **d**. 13
- **e**. 105

Solution: The distance from $\langle 3,4,12 \rangle$ to the origin is $\sqrt{3^2 + 4^2 + 12^2} = 13$. The radius of the sphere is $R = 8$. So the point is 5 units outside the sphere.

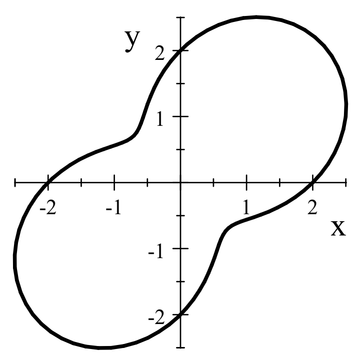
2. Find a and b so that $a(1,2) + b(2,1) = (0,3)$. What is $a + b$?

- **a**. 1 Correct
- **b**. 2
- **c**. 3
- **d**. 4
- **e**. 5

Solution: $a + 2b = 0 \qquad 2a + b = 3$

The 1^{st} equation says $a = -2b$. So the 2^{nd} equation says $-4b + b = 3$ or $b = -1$ and $a = 2$. So $a + b = 1$.

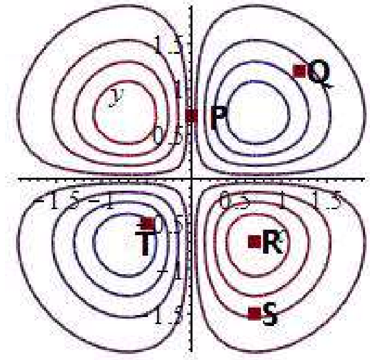
3. The plot at the right is which polar curve?



- **a**. $r = 2 - \cos(2\theta)$
- **b**. $r = 2 + \cos(2\theta)$
- **c**. $r = 2 - \sin(2\theta)$
- **d**. $r = 2 + \sin(2\theta)$ Correct
- **e**. $r = \theta$

Solution: From the plot, when $\theta = 0$, we have $r = 2$, which is only true for equations (c) and (d). When $\theta = \frac{\pi}{4}$, we have $r = 3$, which is only true for equation (d).

4. In the plot at the right, which point could be a local maximum?



a. $P = \left(0, \frac{1}{\sqrt{2}}\right)$

b. $Q = \left(\sqrt{2}, \sqrt{2}\right)$

c. $R = \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$ Correct

d. $S = \left(\frac{1}{\sqrt{2}}, -\sqrt{2}\right)$

e. $T = \left(\frac{-1}{2}, \frac{-1}{2}\right)$

Solution: Near a local maximum, the contours form circles around the local maximum. So R is the local maximum.

5. Find a vector perpendicular to the plane thru the points $P = (2,3,0)$, $Q = (4,-1,-1)$ and $R = (2,0,2)$.

a. $\langle 11,-4,-6\rangle$

b. $\langle -11,3,-2\rangle$

c. $\langle -11,-4,-6\rangle$ Correct

d. $\langle -11,-3,-2\rangle$

e. $\langle -11,4,-6\rangle$

Solution: $\overrightarrow{PQ} = Q - P = \langle 2,-4,-1\rangle$ $\overrightarrow{PR} = R - P = \langle 0,-3,2\rangle$ and

$$\vec{N} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & -4 & -1 \\ 0 & -3 & 2 \end{vmatrix} = \hat{\imath}(-8-3) - \hat{\jmath}(4-0) + \hat{k}(-6) = \langle -11,-4,-6\rangle$$

6. A triangle has vertices at $P = (1,0,4)$, $Q = (1,0,2)$ and $R = \left(2,\sqrt{3},0\right)$. Find the angle at Q.

a. $30°$

b. $45°$

c. $60°$

d. $120°$

e. $135°$ Correct

Solution: $\overrightarrow{QP} = (0,0,2)$ $\overrightarrow{QR} = \left(1,\sqrt{3},-2\right)$ $\left|\overrightarrow{QP}\right| = \sqrt{4} = 2$ $\left|\overrightarrow{QR}\right| = \sqrt{1+3+4} = \sqrt{8} = 2\sqrt{2}$

$$\overrightarrow{QP} \cdot \overrightarrow{QR} = -4 \qquad \cos\theta = \frac{\overrightarrow{QP} \cdot \overrightarrow{QR}}{\left|\overrightarrow{QP}\right|\left|\overrightarrow{QR}\right|} = \frac{-4}{2 \cdot 2\sqrt{2}} = \frac{-1}{\sqrt{2}} \qquad \theta = 135°$$

7. Find the intersection of the line $\dfrac{x-2}{-2} = \dfrac{y-1}{3} = \dfrac{z+2}{1}$ and the plane $2x+y-z=3$. At this point $x+y+z=$

- **a**. 1
- **b**. 3
- **c**. 5 Correct
- **d**. 7
- **e**. 9

Solution: The parametric equation of the line is $(x,y,z) = (2-2t, 1+3t, -2+t)$.
Plug the line into the plane: $2(2-2t)+(1+3t)-(-2+t) = 3$ or $7-2t = 3$ or $-2t = -4$. So $t = 2$, $(x,y,z) = (-2,7,0)$ and $x+y+z = 5$.

8. Find the plane tangent to the graph of the function $z = f(x,y) = x^2\sin(y) + x\cos(y)$ at the point $(x,y) = (2,\pi)$. Its z-intercept is

- **a**. 4π Correct
- **b**. 2π
- **c**. 2
- **d**. -4π
- **e**. -2π

Solution: $f(2,\pi) = 4\sin(\pi) + 2\cos(\pi) = -2$
$f_x(x,y) = 2x\sin(y) + \cos(y)$ $\quad f_x(2,\pi) = 4\sin(\pi) + \cos(\pi) = -1$
$f_y(x,y) = x^2\cos(y) - x\sin(y)$ $\quad f_y(2,\pi) = 4\cos(\pi) - 2\sin(\pi) = -4$
$z = f(2,\pi) + f_x(2,\pi)(x-2) + f_y(2,\pi)(y-\pi) = -2 - 1(x-2) - 4(y-\pi) = -x - 4y + 4\pi$

9. A plane is flying from WEST to EAST, directly over the equator at a constant altitude of 100 kilometers above sea level. (Since the Earth is a sphere, the path of the plane is part of a great circle.) In what direction do $\hat{N}$ and $\hat{B}$ point?

- **a**. $\hat{N}$ points SOUTH and $\hat{B}$ points DOWN
- **b**. $\hat{N}$ points SOUTH and $\hat{B}$ points UP
- **c**. $\hat{N}$ points DOWN and $\hat{B}$ points NORTH Correct
- **d**. $\hat{N}$ points DOWN and $\hat{B}$ points SOUTH
- **e**. $\hat{N}$ points UP and $\hat{B}$ points NORTH

Solution: $\hat{T}$ points EAST. Since the path is a circle the aceleration points toward the center. So $\hat{N}$ points toward the center of the Earth which is DOWN. $\hat{B} = \hat{T} \times \hat{N}$ which points NORTH.

10. Find the mass of a wire in the shape of the semi-circle $\vec{r}(\theta) = (3\cos\theta, 3\sin\theta)$ for $0 \le \theta \le \pi$ if the linear density is given by $\delta = y$.

a. π

b. 3π

c. 6

d. 12

e. 18 Correct

Solution: The tangent vector is $\vec{v} = (-3\sin\theta, 3\cos\theta)$ and its length is $|\vec{v}| = \sqrt{9\sin^2\theta + 9\cos^2\theta} = 3$. The density along the curve is $\delta(\vec{r}(t)) = 3\sin\theta$. So the mass is:

$$M = \int_0^\pi \delta\,ds = \int_0^\pi \delta(\vec{r}(t))\,|\vec{v}|\,d\theta = \int_0^\pi 3\sin\theta\,3\,d\theta = \Big[-9\cos\theta\Big]_0^\pi = 9 - -9 = 18.$$

11. A bead is pushed along a wire in the shape of the twisted cubic $\vec{r}(t) = (t^2, t^3, t)$ by the force $\vec{F} = \langle x, z, -y\rangle$ from $(1,1,1)$ to $(4,8,2)$. Find the work done.

a. 15 Correct

b. 16

c. $\dfrac{45}{2}$

d. 45

e. 48

Solution: $\vec{v} = \langle 2t, 3t^2, 1\rangle \qquad \vec{F}(\vec{r}(t)) = \langle t^2, t, -t^3\rangle \qquad \vec{F}\cdot\vec{v} = 2t^3 + 3t^3 - t^3 = 4t^3$

$$W = \int \vec{F}\cdot d\vec{s} = \int_1^2 \vec{F}\cdot\vec{v}\,dt = \int_1^2 4t^3\,dt = [t^4]_1^2 = 16 - 1 = 15$$

12. Compute $\displaystyle\lim_{h\to 0}\frac{\sin^3(2x+2h+3y) - \sin^3(2x+3y)}{h}$

a. $6\sin^2(2x+3y)\cos(2x+3y)$ Correct

b. $6\cos^2(2x+3y)$

c. $9\sin^2(2x+3y)\cos(2x+3y)$

d. $9\cos^2(2x+3y)$

e. $6\sin^2(2x+3y)$

Solution: This is the definition of the x partial derivative of $\sin^3(2x+3y)$. So we compute $\frac{\partial}{\partial x}\sin^3(2x+3y) = 3\sin^2(2x+3y)\cos(2x+3y)2$

Work Out: (Points indicated. Part credit possible. Show all work.)

13. (20 points) As Duke Skywater flies the Century Eagle through the galaxy he wants to maximize the Power of the Force which is given by $F = \frac{3}{D}$ where D is the dark matter density given by $D = x^2 + y^2 + z^2$. His current position is $\vec{r} = (1,2,2)$.

a. If his current velocity is $\vec{v} = (0.3, 0.5, 0.7)$, what is the current rate of change of the Power of the Force, $\frac{dF}{dt}$? (Plug in numbers but you don't need to simplify.)

Solution: The position says $x = 1, \quad y = 2, \quad z = 2$.

The velocity says $\frac{dx}{dt} = 0.3, \quad \frac{dy}{dt} = 0.5, \quad \frac{dz}{dt} = 0.7$.

Currently $D = x^2 + y^2 + z^2 = 1^2 + 2^2 + 2^2 = 9$.

We use the chain rule twice:

$$\frac{dF}{dt} = \frac{dF}{dD}\frac{dD}{dt} = \frac{dF}{dD}\left(\frac{\partial D}{\partial x}\frac{dx}{dt} + \frac{\partial D}{\partial y}\frac{dy}{dt} + \frac{\partial D}{\partial z}\frac{dz}{dt}\right) = \frac{-3}{D^2}\left(2x\frac{dx}{dt} + 2y\frac{dy}{dt} + 2z\frac{dz}{dt}\right)$$

$$= \frac{-3}{81}(2 \cdot 1 \cdot (0.3) + 2 \cdot 2 \cdot (0.5) + 2 \cdot 2 \cdot (0.7)) = \frac{-5.4}{27} \approx -0.2$$

b. If he wants to change his velocity to increase the Power of the Force as fast as possible, in what **unit** vector direction should he travel?

Solution: The position says $x = 1, \quad y = 2, \quad z = 2$.

Currently $D = x^2 + y^2 + z^2 = 1^2 + 2^2 + 2^2 = 9$.

The Force will increase fastest in the direction of its gradient:

$$\vec{\nabla}F = \left\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right\rangle = \left\langle \frac{dF}{dD}\frac{\partial D}{\partial x}, \frac{dF}{dD}\frac{\partial D}{\partial y}, \frac{dF}{dD}\frac{\partial D}{\partial z} \right\rangle = \frac{dF}{dD}\vec{\nabla}D = \frac{-3}{D^2}\langle 2x, 2y, 2x \rangle = \frac{-3}{81}\langle 2,4,4 \rangle$$

$$\left|\vec{\nabla}F\right| = \frac{3}{81}\sqrt{4 + 16 + 16} = \frac{18}{81} = \frac{2}{9}$$

So the unit vector direction is

$$\frac{1}{\left|\vec{\nabla}F\right|}\vec{\nabla}F = \frac{9}{2}\frac{(-3)}{81}\langle 2,4,4 \rangle = \frac{-1}{6}\langle 2,4,4 \rangle = \left\langle \frac{-1}{3}, \frac{-2}{3}, \frac{-2}{3} \right\rangle$$

14. (20 points) For each limit, prove it exists or does not exist. If it exists, find the limit.

a. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^3}{(x+y^3)^2}$

Solution: Try the straight lines: $\displaystyle\lim_{\substack{y=mx \\ x\to 0}} \frac{xy^3}{(x+y^3)^2} = \lim_{x\to 0} \frac{xm^3x^3}{(x+m^3x^3)^2} = \lim_{x\to 0} \frac{m^3x^2}{(1+m^3x^2)^2} = 0$

So if the limit exists, it must be 0.

Try the cubic $x = y^3$: $\displaystyle\lim_{\substack{x=y^3 \\ y\to 0}} \frac{xy^3}{(x+y^3)^2} = \lim_{y\to 0} \frac{y^6}{(2y^3)^2} = \frac{1}{4}$

Since $\frac{1}{4} \neq 0$, the limit does not exist.

b. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^4+y^4}{x^2+y^2}$

Solution: Try the straight lines: $\displaystyle\lim_{\substack{y=mx \\ x\to 0}} \frac{x^4+y^4}{x^2+y^2} = \lim_{x\to 0} \frac{x^4+m^4x^4}{x^2+m^2x^2} = \lim_{x\to 0} \frac{x^2(1+m^4)}{1+m^2} = 0$

So if the limit exists, it must be 0.

Switch to polar coordinates:

$$\lim_{(x,y)\to(0,0)} \frac{x^4+y^4}{x^2+y^2} = \lim_{\substack{\theta=\theta(r) \\ r\to 0}} \frac{r^4\cos^4\theta + r^4\sin^4\theta}{r^2} = \lim_{r\to 0} r^2(\cos^4\theta + \sin^4\theta)$$

Since $-2 \le \cos^4\theta + \sin^4\theta \le 2$, we have $-r^2 \le r^2(\cos^4\theta + \sin^4\theta) \le r^2$.

By the squeeze theorem, since $\displaystyle\lim_{r\to 0} -r^2 = \lim_{r\to 0} r^2 = 0$, the quantity in the middle also has limit 0.

So $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^4+y^4}{x^2+y^2} = 0$.