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MATH 253 Exam 2 Fall 2016

Sections 201/202 Solutions P. Yasskin

1-8	/40
9	/30
10	/16
11	/15
E.C.	/ 5
Total	/106

Multiple Choice: (5 points each. No part credit.)

1. The function $f = \sin x \cos y$ has a critical point at $(x,y) = \left(\pi, \frac{\pi}{2}\right)$.
Use the Second Derivative Test to classify this critical point.

a. Local Minimum
b. Local Maximum
c. Inflection Point
d. Saddle Point Correct
e. Test Fails

Solution: $f_x = \cos x \cos y \qquad f_y = -\sin x \sin y$

$f_{xx} = -\sin x \cos y \qquad f_{yy} = -\sin x \cos y \qquad f_{xy} = -\cos x \sin y$

$f_{xx}\left(\pi, \frac{\pi}{2}\right) = 0 \qquad f_{yy}\left(\pi, \frac{\pi}{2}\right) = 0 \qquad f_{xy}\left(\pi, \frac{\pi}{2}\right) = 1 \qquad D = f_{xx}f_{yy} - f_{xy}^2 = 0 - 1 = -1$

Saddle Point

2. Find the volume of the solid under $z = 2x^2y$ above the region in the xy-plane between $y = x$ and $y = x^2$.

a. $\frac{1}{12}$
b. $\frac{35}{12}$
c. $\frac{12}{35}$
d. $\frac{1}{35}$
e. $\frac{2}{35}$ Correct

Solution: $V = \iint 2x^2y\,dA = \int_0^1 \int_{x^2}^{x} 2x^2y\,dy\,dx == \int_0^1 \left[x^2y^2\right]_{y=x^2}^{x} dx$

$= \int_0^1 (x^4 - x^6)\,dx = \left[\frac{x^5}{5} - \frac{x^7}{7}\right]_{x=0}^{1} = \frac{1}{5} - \frac{1}{7} = \frac{7-5}{35} = \frac{2}{35}$

3. Compute $\iint \sin(x^2)\,dx\,dy$ over the triangle with vertices $(0,0)$, $(\sqrt{\pi},0)$, $(\sqrt{\pi},\sqrt{\pi})$.

a. $-\pi$

b. $-\sqrt{\pi}$

c. 1 Correct

d. $\sqrt{\pi}$

e. π

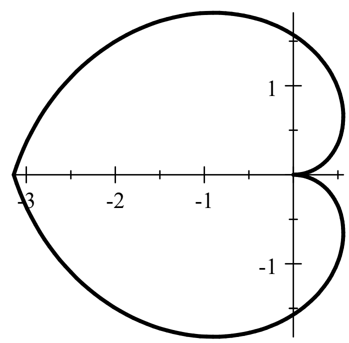
Solution: You must do the y-integral first because you don't know the antiderivative of $\sin(x^2)$.

The edges are $y = 0$, $x = \sqrt{\pi}$, $y = x$.

$$\iint \sin(x^2)\,dx\,dy = \int_0^{\sqrt{\pi}}\int_0^x \sin(x^2)\,dy\,dx = \int_0^{\sqrt{\pi}}\Big[y\sin(x^2)\Big]_{y=0}^{x} dx = \int_0^{\sqrt{\pi}} x\sin(x^2)\,dx$$

$$= \left[\frac{-1}{2}\cos(x^2)\right]_{x=0}^{\sqrt{\pi}} = \frac{1}{2} - -\frac{1}{2} = 1$$

4. Find the area of the heart shaped region inside the polar curve $r = |\theta|$.



a. $\frac{\pi^3}{6}$

b. $\frac{\pi^3}{3}$ Correct

c. $\frac{4\pi^3}{3}$

d. $\frac{8\pi^3}{3}$

e. $\frac{16\pi^3}{3}$

Solution: Double the upper half:

$$A = 2\iint 1\,dA = 2\int_0^{\pi}\int_0^{\theta} r\,dr\,d\theta = 2\int_0^{\pi}\left[\frac{r^2}{2}\right]_{r=0}^{\theta} d\theta = 2\int_0^{\pi}\left(\frac{\theta^2}{2}\right) d\theta = 2\left[\frac{\theta^3}{6}\right]_{\theta=0}^{\pi} = \frac{\pi^3}{3}$$

5. The solid half cylinder $0 \le y \le \sqrt{9 - x^2}$ with $0 \le z \le 2$ has density $\delta = y$. Find the total mass.

a. 9

b. 18

c. 36 Correct

d. 9π

e. 18π

In cylindrical coordinates, $\delta = y = r\sin\theta$ and $J = r$.

$$M = \iiint \delta\,dV = \int_0^2\int_0^{\pi}\int_0^3 r\sin\theta\, r\,dr\,d\theta\,dz = 2\left[\frac{r^3}{3}\right]_{r=0}^{3}\Big[-\cos\theta\Big]_{\theta=0}^{\pi} = 36$$

6. The solid half cylinder $0 \le y \le \sqrt{9 - x^2}$ with $0 \le z \le 2$ has density $\delta = y$.
Find the y-component of the center of mass.

a. $\dfrac{9\pi}{16}$ Correct

b. $\dfrac{9\pi}{4}$

c. 9π

d. $\dfrac{81\pi}{4}$

e. $\dfrac{\pi}{2}$

Solution: $M_y = \iiint y\delta\, dV = \int_0^2 \int_0^\pi \int_0^3 r^2 \sin^2\theta\, r\, dr\, d\theta\, dz = 2\left[\dfrac{r^4}{4}\right]_{r=0}^3 \int_0^\pi \dfrac{1-\cos 2\theta}{2}\, d\theta$

$$= \frac{81}{2}\left[\frac{1}{2}\left(\theta - \frac{\sin 2\theta}{2}\right)\right]_0^\pi = \frac{81\pi}{4}$$

$$\bar{y} = \frac{M_y}{M} = \frac{81\pi}{4 \cdot 36} = \frac{9\pi}{16}$$

7. Compute $\iiint \nabla \cdot \vec{F}\, dV$ on the solid hemisphere $0 \le z \le \sqrt{25 - x^2 - y^2}$
for the vector field $\vec{F} = \left(\frac{2}{3}x^3, \frac{2}{3}y^3, z(x^2 + y^2 + z^2)\right)$.

a. 0

b. $5^3\pi^2$

c. $\dfrac{14}{3}\pi 5^4$

d. $6\pi 5^4$ Correct

e. $12\pi 5^4$

Solution: $\nabla \cdot \vec{F} = 2x^2 + 2y^2 + x^2 + y^2 + 3z^2 = 3(x^2 + y^2 + z^2) = 3\rho^2$

$$\iiint \nabla \cdot \vec{F}\, dV = \int_0^{2\pi} \int_0^{\pi/2} \int_0^5 3\rho^2 \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta = 2\pi\left[-\cos\varphi\right]_{\varphi=0}^{\pi/2}\left[\frac{3\rho^5}{5}\right]_{\rho=0}^5 = 6\pi 5^4$$

8. If $\vec{F} = (-yz, xz, z^2)$, compute $\vec{F} \cdot \vec{\nabla} \times \vec{F}$.

a. z^3

b. $z^3 - xyz$

c. $2z^3$ Correct

d. $2z^3 - 2xyz$

e. 0

Solution: $\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ -yz & xz & z^2 \end{vmatrix} = \hat{\imath}(x) - \hat{\jmath}(-y) + \hat{k}(z - -z) = (x, y, 2z)$

$$\vec{F} \cdot \vec{\nabla} \times \vec{F} = -yzx + xzy + z^2 2z = 2z^3$$

Work Out: (20 points each. Part credit possible. Show all work.)

9. (30 points) Consider the hemispherical surface

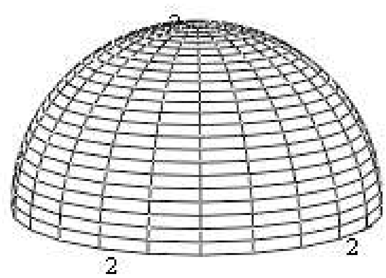
$$z = \sqrt{4 - x^2 - y^2}$$

which may be parametrized by

$$\vec{R}(\varphi,\theta) = (2\sin\varphi\cos\theta, 2\sin\varphi\sin\theta, 2\cos\varphi)$$

and the vector field $\vec{F} = (-y^3, x^3, z(x^2+y^2))$.

Find each of the following:



a. (2 pts) $\vec{e}_\varphi = (2\cos\varphi\cos\theta, \quad 2\cos\varphi\sin\theta, \quad -2\sin\varphi)$

b. (2 pts) $\vec{e}_\theta = (-2\sin\varphi\sin\theta, \quad 2\sin\varphi\cos\theta, \quad 0)$

c. (3 pts) $\vec{N} = \hat{\imath}(4\sin^2\varphi\cos\theta) - \hat{\jmath}(-4\sin^2\varphi\sin\theta) + \hat{k}(4\sin\varphi\cos\varphi\cos^2\theta + 4\sin\varphi\cos\varphi\sin^2\theta)$

$$= (4\sin^2\varphi\cos\theta, 4\sin^2\varphi\sin\theta, 4\sin\varphi\cos\varphi)$$

d. (2 pts) $\left|\vec{N}\right| = \sqrt{16\sin^4\varphi\cos^2\theta + 16\sin^4\varphi\sin^2\theta + 16\sin^2\varphi\cos^2\varphi}$

$$= 4\sqrt{\sin^4\varphi + \sin^2\varphi\cos^2\varphi} = 4\sin\varphi$$

e. (5 pts) The total mass of the surface if the surface density is $\delta = z$.

$$M = \iint \delta\, dS = \iint z\left|\vec{N}\right| d\varphi\, d\theta = \int_0^{2\pi}\int_0^{\pi/2} 2\cos\varphi\, 4\sin\varphi\, d\varphi\, d\theta = 2\pi \cdot 8 \cdot \left.\frac{\sin^2\varphi}{2}\right|_{\varphi=0}^{\pi/2} = 8\pi$$

f. (6 pts) The z-component of the center of mass of the surface if the surface density is $\delta = z$.

$$M_z = \iint z\delta\, dS = \iint z^2\left|\vec{N}\right| d\varphi\, d\theta = \int_0^{2\pi}\int_0^{\pi/2} 4\cos^2\varphi\, 4\sin\varphi\, d\varphi\, d\theta = 2\pi \cdot 16 \cdot \left.\frac{-\cos^3\varphi}{3}\right|_{\varphi=0}^{\pi/2} = \frac{32\pi}{3}$$

$$\bar{z} = \frac{M_z}{M} = \frac{32\pi}{3 \cdot 8\pi} = \frac{4}{3}$$

g. (3 pts) $\nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ -y^3 & x^3 & (x^2+y^2)z \end{vmatrix}$

$$= \hat{\imath}(2yz) - \hat{\jmath}(2xz) + \hat{k}(3x^2 + 3y^2) = (2yz, -2xz, 3x^2 + 3y^2)$$

(continued)

h. (2 pts) $\nabla \times \vec{F}\left(\vec{R}(\varphi,\theta)\right) =$

$$= (8\sin\varphi\cos\varphi\sin\theta, -8\sin\varphi\cos\varphi\cos\theta, 3\cdot 4\sin^2\varphi\cos^2\theta + 3\cdot 4\sin^2\varphi\sin^2\theta)$$
$$= (8\sin\varphi\cos\varphi\sin\theta, -8\sin\varphi\cos\varphi\cos\theta, 12\sin^2\varphi)$$

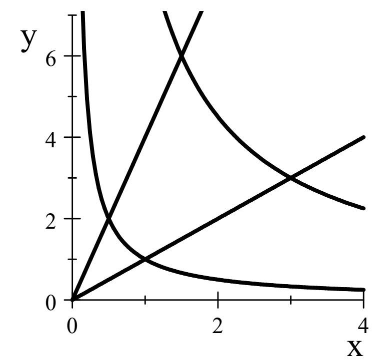
i. (5 pts) $\iint \nabla \times \vec{F} \cdot d\vec{S}$ with normal pointing up.

From problem 9, $\vec{N} = (4\sin^2\varphi\cos\theta, 4\sin^2\varphi\sin\theta, 4\sin\varphi\cos\varphi)$ which points up.

$$\iint \nabla \times \vec{F} \cdot d\vec{S} = \iint \nabla \times \vec{F} \cdot \vec{N}\, d\varphi\, d\theta$$
$$= \iint (32\sin^3\varphi\cos\varphi\sin\theta\cos\theta - 32\sin^3\varphi\cos\varphi\sin\theta\cos\theta + 48\sin^3\varphi\cos\varphi)\, d\varphi\, d\theta$$
$$= 48\int_0^{2\pi}\int_0^{\pi/2} \sin^3\varphi\cos\varphi\, d\varphi\, d\theta = 48\cdot 2\pi\left[\frac{\sin^4\varphi}{4}\right]_0^{\pi/2} = 24\pi$$

10. (16 points) Compute $\iint_R y^2\, dx\, dy$ over the diamond shaped region R bounded by

$$y = \frac{1}{x}, \qquad y = \frac{9}{x}, \qquad y = x, \qquad y = 4x$$



FULL CREDIT for integrating in the curvilinear coordinates (u,v) where $u^2 = xy$ and $v^2 = \frac{y}{x}$.

HALF CREDIT for integrating in rectangular coordinates.

Solution: $\left\{\begin{array}{l} u^2 = xy \\ v^2 = \frac{y}{x} \end{array}\right\} \Rightarrow \left\{\begin{array}{l} u^2v^2 = y^2 \\ \frac{u^2}{v^2} = x^2 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} x = \frac{u}{v} \\ y = uv \end{array}\right\}$

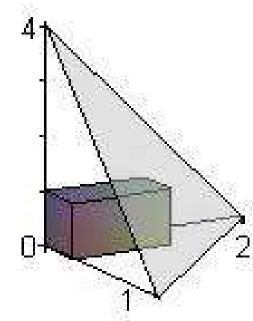
$$J = \left|\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\right| = \left|\left|\begin{array}{cc} \frac{1}{v} & -\frac{u}{v^2} \\ v & u \end{array}\right|\right| = \left|\frac{u}{v} - -\frac{u}{v}\right| = \frac{2u}{v}$$

$xy = 1 \Rightarrow u^2 = 1 \Rightarrow u = 1$ $xy = 9 \Rightarrow u^2 = 9 \Rightarrow u = 3$ So: $1 \le u \le 3$

$\frac{y}{x} = 1 \Rightarrow v^2 = 1 \Rightarrow v = 1$ $\frac{y}{x} = 4 \Rightarrow v^2 = 4 \Rightarrow v = 2$ So: $1 \le v \le 2$

$$\iint_R y^2\, dx\, dy = \int_1^2\int_1^3 u^2v^2\, \frac{2u}{v}\, du\, dv = 2\int_1^2\int_1^3 u^3 v\, du\, dv$$
$$= 2\left[\frac{u^4}{4}\right]_{u=1}^3\left[\frac{v^2}{2}\right]_{v=1}^2 = 2\left[\frac{81}{4} - \frac{1}{4}\right]\left[\frac{4}{2} - \frac{1}{2}\right] = 60$$

11. (15 points) Find the volume of the largest rectangular box in the first octant with three faces in the coordinate planes and one vertex on the plane $x + \frac{y}{2} + \frac{z}{4} = 1$.



Solve either by Eliminating a Variable or by Lagrange Multipliers.
5 points extra credit for doing both. Clearly separate solutions.

Solution by Eliminating a Variable: $x = 1 - \frac{y}{2} - \frac{z}{4}$

$V = xyz = \left(1 - \frac{y}{2} - \frac{z}{4}\right)yz = yz - \frac{1}{2}y^2z - \frac{1}{4}yz^2$

$V_y = z - yz - \frac{1}{4}z^2 = z\left(1 - y - \frac{1}{4}z\right) = 0 \qquad V_z = y - \frac{1}{2}y^2 - \frac{1}{2}yz = y\left(1 - \frac{1}{2}y - \frac{1}{2}z\right) = 0$

If y or z is 0, then the volume is 0 and this cannot be the maximum volume.

So we solve $1 - y - \frac{1}{4}z = 0$ and $1 - \frac{1}{2}y - \frac{1}{2}z = 0$ or $y + \frac{1}{4}z = 1$ and $y + z = 2$.

Subtract: $\frac{3}{4}z = 1 \qquad z = \frac{4}{3} \qquad y = 2 - z = \frac{2}{3} \qquad x = 1 - \frac{y}{2} - \frac{z}{4} = 1 - \frac{1}{3} - \frac{1}{3} = \frac{1}{3}$

$V = xyz = \frac{1}{3} \cdot \frac{2}{3} \cdot \frac{4}{3} = \frac{8}{27}$

Solution by Lagrange Multipliers: $V = xyz \qquad g = x + \frac{y}{2} + \frac{z}{4}$

$\vec{\nabla} V = (yz, xz, xy) \qquad \vec{\nabla} g = \left(1, \frac{1}{2}, \frac{1}{4}\right) \qquad \vec{\nabla} V = \lambda \vec{\nabla} g$:

$yz = \lambda \qquad xz = \frac{1}{2}\lambda \qquad xy = \frac{1}{4}\lambda \qquad \Rightarrow \qquad xz = \frac{1}{2}yz \qquad xy = \frac{1}{4}yz$

If y or z is 0, then the volume is 0 and this cannot be the maximum volume.

So $x = \frac{1}{2}y \qquad x = \frac{1}{4}z$ or $y = 2x \qquad z = 4x$

From the constraint: $x + \frac{2x}{2} + \frac{4x}{4} = 1$ or $x = \frac{1}{3} \qquad y = \frac{2}{3} \qquad z = \frac{4}{3}$

$V = xyz = \frac{1}{3} \cdot \frac{2}{3} \cdot \frac{4}{3} = \frac{8}{27}$