MATH 253 Honors | EXAM 1 | Spring 1999

Sections 201-203 | Solutions | P. Yasskin

Multiple Choice: (5 points each)

Problems 1 – 4: Consider the vectors: $\vec{a} = (2,0,-2)$, $\vec{b} = (0,-3,3)$ and $\vec{c} = (1,1,1)$.

1. The angle between $\vec{a}$ and $\vec{b}$ is

a. $30°$

b. $45°$

c. $90°$

d. $120°$ correctchoice

e. $150°$

$$\cos\theta = \frac{\vec{a}\bullet\vec{b}}{|\vec{a}|\,|\vec{b}|} = \frac{-6}{\sqrt{8}\sqrt{18}} = -\frac{1}{2} \qquad \theta = 120°$$

2. The vector projection of $\vec{a}$ along $\vec{b}$ is

a. $(0,1,-1)$ correctchoice

b. $(0,-1,1)$

c. $(-2,-3,5)$

d. $\left(0,3\sqrt{2},-3\sqrt{2}\right)$

e. $\left(0,-3\sqrt{2},3\sqrt{2}\right)$

$$\text{proj}_{\vec{b}}\vec{a} = \frac{\vec{a}\bullet\vec{b}}{|\vec{b}|^2}\vec{b} = \frac{-6}{18}(0,-3,3) = (0,1,-1)$$

3. The area of a triangle with $\vec{a}$ and $\vec{b}$ as two sides is

a. 3

b. 6

c. $3\sqrt{3}$ correctchoice

d. $6\sqrt{3}$

e. 54

$$\vec{a}\times\vec{b} = \begin{vmatrix} i & j & k \\ 2 & 0 & -2 \\ 0 & -3 & 3 \end{vmatrix} = i(-6) - j(6) + k(-6) = (-6,-6,-6) \qquad A = \frac{1}{2}\left|\vec{a}\times\vec{b}\right| = 3\sqrt{3}$$

4. The volume of the parallelepiped with edges $\vec{a}$, $\vec{b}$ and $\vec{c}$ is

a. -18
b. -6
c. 3
d. 6
e. 18 correctchoice

$V = \left|\vec{a} \times \vec{b} \bullet \vec{c}\right| = |(-6,-6,-6) \bullet (1,1,1)| = |-18| = 18$ Volume cannot be negative!

Problems 5 – 7: The pressure in an ideal gas is given by $P = k\rho T$ where k is a constant, ρ is the density and T is the temperature. The pressure, density and temperature are all functions of position. At the point $Q = (1,2,3)$, the density is $\rho(Q) = 1.5$ and its gradient is $\vec{\nabla}\rho(Q) = (.2,.3,-.1)$. Also at that point, the temperature is $T(Q) = 24$ and its gradient is $\vec{\nabla}T(Q) = (-3,1,2)$.

5. At the point Q, the pressure is $P(Q) = 36k$. What is the gradient of the pressure?

a. $\vec{\nabla}P(Q) = k(.7,-.1,1.1)$
b. $\vec{\nabla}P(Q) = k(.3,8.7,.6)$ correctchoice
c. $\vec{\nabla}P(Q) = k(.3,-8.7,.6)$
d. $\vec{\nabla}P(Q) = k(-2.8,1.3,1.9)$
e. $\vec{\nabla}P(Q) = k(-2.8,-1.3,1.9)$

Apply the product rule to $P = k\rho T$: (Or do each partial derivative separately.)

$$\vec{\nabla}P(Q) = kT\vec{\nabla}\rho(Q) + k\rho\vec{\nabla}T(Q) = k24(.2,.3,-.1) + k1.5(-3,1,2)$$
$$= k(4.8,7.2,-2.4) + k(-4.5,1.5,3) = k(.3,8.7,.6)$$

6. If a fly is located at the point Q, in what direction should the fly travel to **cool off** as soon as possible?

a. $(-.2,-.3,.1)$
b. $(3,-1,-2)$ correctchoice
c. $(-3,1,2)$
d. $(2,-1,3)$
e. $(2,1,3)$

The temperature decreases fastest in the direction of $-\vec{\nabla}T(Q) = -(-3,1,2) = (3,-1,-2)$

7. If a fly is located at the point Q and travelling with velocity $\vec{v} = (3,4,12)$, how fast is the density changing at the location of the fly?

a. $\frac{d\rho}{dt}(Q) = -7.8$

b. $\frac{d\rho}{dt}(Q) = -.6$

c. $\frac{d\rho}{dt}(Q) = \frac{.6}{13}$

d. $\frac{d\rho}{dt}(Q) = .6$ correctchoice

e. $\frac{d\rho}{dt}(Q) = 7.8$

$$\frac{d\rho}{dt}(Q) = \vec{v} \bullet \vec{\nabla}\rho(Q) = (3,4,12) \bullet (.2,.3,-.1) = .6 + 1.2 - 1.2 = .6$$

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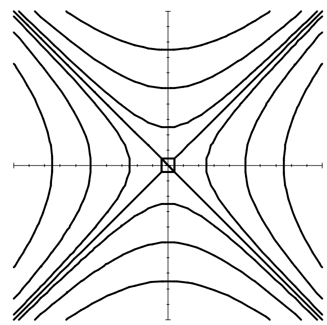
8. The graph at the right is the contour plot of which function?



a. $y^2 - x^2$ correctchoice

b. xy

c. $x^2 + y^2$

d. $y - x^2$

e. $x - y^2$

$y^2 - x^2 = C$ is a hyperbola which opens up and down if $C > 0$ and left and right if $C < 0$.

9. (24 points) Consider the parametric curve $\vec{r}(t) = (t^3, 3t^2, 6t)$.

a. Compute the velocity and acceleration:

$\vec{v} = (3t^2, 6t, 6)$

$\vec{a} = (6t, 6, 0)$

b. Find a parametric equation for the line tangent to the curve at $t = 1$.

$X = P + t\vec{v} \qquad X = (x,y,z) \qquad P = \vec{r}(1) = (1,3,6) \qquad \vec{v}(1) = (3,6,6)$
$(x,y,z) = (1,3,6) + t(3,6,6) = (1+3t, 3+6t, 6+6t)$

c. Find a non-parametric (symmetric) equation for the line tangent to the curve at $t = 1$.

$x = 1 + 3t \qquad y = 3 + 6t \qquad z = 6 + 6t$
$t = \dfrac{x-1}{3} = \dfrac{y-3}{6} = \dfrac{z-6}{6}$

d. Find a parametric equation for the plane instantaneously containing the curve at $t = 1$.

$X = P + t\vec{v} + s\vec{a}$
$X = (x,y,z) \qquad P = \vec{r}(1) = (1,3,6) \qquad \vec{v}(1) = (3,6,6) \qquad \vec{a}(1) = (6,6,0)$
$(x,y,z) = (1,3,6) + t(3,6,6) + s(6,6,0) = (1+3t+6s, 3+6t+6s, 6+6t)$

e. Find a non-parametric equation for the plane instantaneously containing the curve at $t = 1$.

$$\vec{N} = \vec{v} \times \vec{a} = \begin{vmatrix} i & j & k \\ 3 & 6 & 6 \\ 6 & 6 & 0 \end{vmatrix} = i(-36) - j(-36) + k(18-36) = (-36, 36, -18)$$

$\vec{N} \bullet X = \vec{N} \bullet P \qquad -36x + 36y - 18z = -36 \bullet 1 + 36 \bullet 3 - 18 \bullet 6 = -36 \qquad 2x - 2y + z = 2$

f. Find the arclength of the curve between $t = 0$ and $t = 2$.

$|\vec{v}| = |(3t^2, 6t, 6)| = \sqrt{9t^4 + 36t^2 + 36} = \sqrt{9(t^2+2)^2} = 3(t^2+2) = 3t^2 + 6$

$L = \int_0^2 |\vec{v}|\,dt = \int_0^2 3t^2 + 6\,dt = \left[t^3 + 6t\right]_0^2 = 8 + 12 = 20$

10. (10 points) Does each limit exist? Why or why not? Find the value of the one that exists. (***Up to 4 points extra credit for a good explanation***.)

a. $\lim\limits_{(x,y)\to(0,0)} \dfrac{2xy}{x^2 + 2y^2}$

Look at the path $y = mx$:

$$\lim_{\substack{y=mx \\ x\to 0}} \frac{2xy}{x^2 + 2y^2} = \lim_{x\to 0} \frac{2xmx}{x^2 + 2m^2x^2} = \lim_{x\to 0} \frac{2m}{1 + 2m^2} = \frac{2m}{1 + 2m^2}$$

This depends on direction. So the limit does not exist.

b. $\lim\limits_{(x,y)\to(0,0)} \dfrac{2xy}{\sqrt{x^2 + 2y^2}}$

Look at the path $y = mx$:

$$\lim_{\substack{y=mx \\ x\to 0}} \frac{2xy}{\sqrt{x^2 + 2y^2}} = \lim_{x\to 0} \frac{2xmx}{\sqrt{x^2 + 2m^2x^2}} = \lim_{x\to 0} \frac{2xm}{\sqrt{1 + 2m^2}} = 0$$

This is independent of direction. So we expect the limit exists.

To prove it, we look at the path $(x,y) = (r\cos\theta(r), r\sin\theta(r))$

$$\lim_{(x,y)\to(0,0)} \frac{2xy}{\sqrt{x^2 + 2y^2}} = \lim_{r\to 0} \frac{2r^2\cos\theta\sin\theta}{\sqrt{r^2\cos^2\theta + 2r^2\sin^2\theta}} = \lim_{r\to 0} 2r\frac{\cos\theta\sin\theta}{\sqrt{\cos^2\theta + 2\sin^2\theta}}$$

$$= \lim_{r\to 0} 2r\frac{\cos\theta\sin\theta}{\sqrt{1 + \sin^2\theta}} = 0$$

because $\dfrac{\cos\theta\sin\theta}{\sqrt{1 + \sin^2\theta}}$ is bounded since

i. the numerator is bounded: $|\cos\theta\sin\theta| < 1$ and

ii. the denominator cannot go to zero: $1 + \sin^2\theta > 1$

11. (13 points) Find the equation of the plane tangent to the graph of the function $f(x,y) = 3x\sin y - 2y\cos x$ at the point $(x,y) = \left(0, \frac{\pi}{2}\right)$.

$$f(x,y) = 3x\sin y - 2y\cos x \qquad f\left(0, \tfrac{\pi}{2}\right) = 0 - 2\tfrac{\pi}{2}\cos 0 = -\pi$$

$$f_x(x,y) = 3\sin y + 2y\sin x \qquad f_x\left(0, \tfrac{\pi}{2}\right) = 3\sin\tfrac{\pi}{2} + 0 = 3$$

$$f_y(x,y) = 3x\cos y - 2\cos x \qquad f_y\left(0, \tfrac{\pi}{2}\right) = 0 - 2\cos 0 = -2$$

$$z = f\left(0, \tfrac{\pi}{2}\right) + f_x\left(0, \tfrac{\pi}{2}\right)(x - 0) + f_y\left(0, \tfrac{\pi}{2}\right)\left(y - \tfrac{\pi}{2}\right)$$

$$= -\pi + 3(x) - 2\left(y - \tfrac{\pi}{2}\right) = -\pi + 3x - 2y + \pi = 3x - 2y$$

$z = -\pi + 3(x) - 2\left(y - \frac{\pi}{2}\right)$ or $z = 3x - 2y$

12. (13 points) Find the equation of the plane tangent to the surface $F(x,y,z) = x^2y + y^3z + z^4x = 29$ at the point $P = (x,y,z) = (3,2,1)$.

$$\vec{\nabla}F = (2xy + z^4, x^2 + 3y^2z, y^3 + 4z^3x)$$

$$\vec{N} = \vec{\nabla}F\Big|_{(3,2,1)} = (2 \bullet 3 \bullet 2 + 1^4, 3^2 + 3 \bullet 2^2 \bullet 1, 2^3 + 4 \bullet 1^3 \bullet 3) = (13, 21, 20)$$

$$\vec{N} \bullet X = \vec{N} \bullet P \qquad (13,21,20) \bullet (x,y,z) = (13,21,20) \bullet (3,2,1) = 39 + 42 + 20 = 101$$

$$13x + 21y + 20z = 101$$