MATH 253 Honors EXAM 3 Spring 1999

Sections 201-203 Solutions P. Yasskin

1. If $F = (xz, yx, zy)$ then $\vec{\nabla} \bullet \vec{F} =$

- **a**. $z - x + y$
- **b**. $(z, -x, y)$
- **c**. $x + y + z$ correctchoice
- **d**. (z, x, y)
- **e**. $-x + y - z$

$$\vec{\nabla} \bullet \vec{F} = \frac{\partial(xz)}{\partial x} + \frac{\partial(yx)}{\partial y} + \frac{\partial(zy)}{\partial z} = z + x + y$$

2. If $F = (xz, yx, zy)$ then $\vec{\nabla} \times \vec{F} =$

- **a**. $z - x + y$
- **b**. $(z, -x, y)$
- **c**. $x + y + z$
- **d**. (z, x, y) correctchoice
- **e**. $-x + y - z$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ xz & yx & zy \end{vmatrix} = i\left(\frac{\partial(zy)}{\partial y} - \frac{\partial(yx)}{\partial z}\right) - j\left(\frac{\partial(zy)}{\partial x} - \frac{\partial(xz)}{\partial z}\right) + k\left(\frac{\partial(yx)}{\partial x} - \frac{\partial(xz)}{\partial y}\right)$$

$$= (z, x, y)$$

3. If $F = (\frac{x \sin z}{x^2 + y^2}, \frac{y \cos x}{x^2 + y^2}, \tan(zy))$ then $\vec{\nabla} \bullet \vec{\nabla} \times \vec{F} =$

- **a**. $\frac{\sin z - \cos x}{(x^2 + y^2)^2} + 2\sec(zy)\tan(zy))$
- **b**. $\frac{\sin z - \cos x}{(x^2 + y^2)^2} + 2\sec(zy)\tan(zy))$
- **c**. $\frac{\sin z - \cos x}{(x^2 + y^2)^3} + 2y\sec(zy)\tan(zy)) + \sec^2(zy)$
- **d**. $\frac{\cos z - \sin x}{(x^2 + y^2)^3} + 2y\sec(zy)\tan(zy)) + \sec^2(zy)$
- **e**. None of These correctchoice

The divergence of a curl is always zero.

4. Compute the line integral $\int y\,dx - x\,dy$ counterclockwise around the semicircle $x^2 + y^2 = 9$ from $(3,0)$ to $(-3,0)$. (HINT: Parametrize the curve.)

a. -9π correctchoice

b. -2π

c. π

d. 2π

e. 9π

$\vec{r}(t) = (3\cos t, 3\sin t)$ $\quad \vec{v}(t) = (-3\sin t, 3\cos t)$ $\quad \vec{F} = (y, -x) = (3\sin t, -3\cos t)$

$$\int y\,dx - x\,dy = \int \vec{F} \bullet d\vec{s} = \int \vec{F} \bullet \vec{v}\,dt = \int_0^{\pi} \left(-9\sin^2 t - 9\cos^2 t\right)\,dt = -9\int_0^{\pi} 1\,dt = -9\pi$$

5. Compute the line integral $\int \vec{F} \bullet d\vec{s}$ for the vector field $\vec{F} = \left(\frac{1}{x}, \frac{1}{y}\right)$ along the curve $\vec{r}(t) = \left(e^{\cos(t^2)}, e^{\sin(t^2)}\right)$ for $0 \le t \le \sqrt{\pi}$. (HINT: Find a potential.)

a. -2 correctchoice

b. 0

c. $\frac{2}{e}$

d. 1

e. π

$\vec{F} = \left(\frac{1}{x}, \frac{1}{y}\right) = \vec{\nabla} f$ where $f = \ln x + \ln y$. By the F.T.C. for curves:

The endpoints are $A = (e^{\cos(0)}, e^{\sin(0)}) = (e, 1)$ and $B = (e^{\cos(\pi)}, e^{\sin(\pi)}) = (e^{-1}, 1)$.

$$\int \vec{F} \bullet d\vec{s} = \int_A^B \vec{\nabla} f \bullet d\vec{s} = f(B) - f(A) = [\ln(e^{-1}) - \ln 1] - [\ln e - \ln 1] = -1 - 1 = -2$$

6. Compute $\iint_{\partial C} \vec{F} \bullet d\vec{S}$ for the vector field $\vec{F} = \left(zx^3, zy^3, z^2(x^2 + y^2)\right)$ over the complete surface of the solid cylinder $C = \left\{(x,y,z) \mid x^2 + y^2 \le 4,\ 0 \le z \le 3\right\}$ with normal pointing outward.

a. 360π

b. 180π correctchoice

c. $90\pi 1$

d. 60π

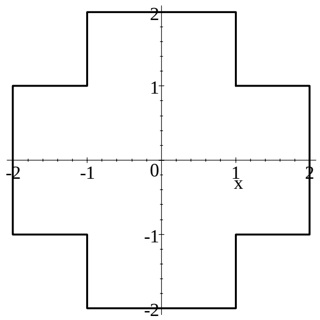
e. 30π

$\vec{\nabla} \bullet \vec{F} = 3zx^2 + 3zy^2 + 2z(x^2 + y^2) = 5z(x^2 + y^2)$. By Gauss' Theorem, using cylindrical coordinates

$$\iint_{\partial C} \vec{F} \bullet d\vec{S} = \iiint_C \vec{\nabla} \bullet \vec{F}\,dV = \int_0^3 \int_0^{2\pi} \int_0^2 5z(r^2)\,r\,dr\,d\theta\,dz = 5\left[\frac{z^2}{2}\right]_0^3 [2\pi] \left[\frac{r^4}{4}\right]_0^2$$

$$= 5\left[\frac{9}{2}\right][2\pi][4] = 180\pi$$

7. (20 points) Compute the line integral $\oint \frac{y}{x^2+y^2}\ dx - \frac{x}{x^2+y^2}\ dy$ counterclockwise around the boundary of the plus sign shown below.
Be sure to justify any theorem you use. (Hint: The answer is not zero.)



$P = \frac{y}{x^2+y^2} \qquad Q = \frac{-x}{x^2+y^2}$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{(x^2+y^2)(-1)-(-x)(2x)}{(x^2+y^2)^2} - \frac{(x^2+y^2)(1)-(y)(2y)}{(x^2+y^2)^2}$$
$$= \frac{x^2-y^2}{(x^2+y^2)^2} - \frac{x^2-y^2}{(x^2+y^2)^2} = 0$$

So P and Q are defined everywhere but the origin and $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$ everywhere but the origin. So by Green's Theorem, we can change the path provided the origin is not in the region between the paths. We take the new path to be the circle $x^2+y^2 = 1$.

$\vec{r}(t) = (\cos t, \sin t) \qquad \vec{v} = (-\sin t, \cos t) \qquad \vec{F} = (P,Q) = (\sin t, -\cos t) \qquad \vec{F}\bullet\vec{v} = -1$

$$\oint \frac{y}{x^2+y^2}\ dx - \frac{x}{x^2+y^2}\ dy = \int_0^{2\pi} \vec{F}\bullet\vec{v}\ dt = \int_0^{2\pi} -1\ dt = -2\pi$$

8. (30 points) Stokes' Theorem states that if S is a surface in 3-space and ∂S is its boundary curve traversed counterclockwise as seen from the tip of the normal to S then

$$\iint_S \vec{\nabla} \times \vec{F} \bullet d\vec{S} = \oint_{\partial S} \vec{F} \bullet d\vec{s}$$

Verify Stokes' Theorem if $F = (yx^2, -xy^2, x^2z + y^2z)$ and S is the hemisphere $z = \sqrt{4 - x^2 - y^2}$ with **normal pointing up and out**.

8a. (10 points) Compute $\oint_{\partial S} \vec{F} \bullet d\vec{s}$ using the following steps: (Remember to check the orientation of the curve.)

$$\vec{r}(\theta) = (2\cos\theta, 2\sin\theta, 0)$$
$$\vec{v}(\theta) = (-2\sin\theta, 2\cos\theta, 0)$$
$$\vec{F}\left(\vec{r}(\theta)\right) = \left(8\sin\theta\cos^2\theta, -8\cos\theta\sin^2\theta, 0\right)$$
$$\oint_{\partial S} \vec{F} \bullet d\vec{s} = \oint_{\partial S} \vec{F} \bullet \vec{v}\ d\theta = \int_0^{2\pi} -16\sin^2\theta\cos^2\theta - 16\cos^2\theta\sin^2\theta\ d\theta = -32\int_0^{2\pi} \sin^2\theta\cos^2\theta\ d\theta$$
$$= -8\int_0^{2\pi} \sin^2(2\theta)\ d\theta = -8\left[\frac{1}{2}(2\pi)\right] = -8\pi$$

8b. (5 points) Compute $\vec{\nabla} \times \vec{F}$. (HINT: Use rectangular coordinates.)

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ yx^2 & -xy^2 & x^2z + y^2z \end{vmatrix} = i(2yz) - j(2xz) + k(-y^2 - x^2) = (2yz, -2xz, -x^2 - y^2)$$

8c. (15 points) Compute $\iint_S \vec{\nabla} \times \vec{F} \bullet d\vec{S}$ using the following steps:

Recall $F = (yx^2, -xy^2, x^2z + y^2z)$ and S is the hemisphere $z = \sqrt{4 - x^2 - y^2}$ with **normal pointing up and out**.

$$\vec{R}(\theta,\phi) = (2\sin\phi\cos\theta, 2\sin\phi\sin\theta, 2\cos\phi)$$
$$\vec{R}_\theta = (-2\sin\phi\sin\theta,\quad 2\sin\phi\cos\theta,\quad 0\quad)$$
$$\vec{R}_\phi = (2\cos\phi\cos\theta,\quad 2\cos\phi\sin\theta,\quad -2\sin\phi)$$
$$\vec{N} = i\left(-4\sin^2\phi\cos\theta\right) - j\left(4\sin^2\phi\sin\theta\right) + k\left(-4\sin\phi\cos\phi\sin^2\theta - 4\sin\phi\cos\phi\cos^2\theta\right)$$

Reverse $\vec{N}$:

$$\vec{N} = \left(4\sin^2\phi\cos\theta, 4\sin^2\phi\sin\theta, 4\sin\phi\cos\phi\right)$$
$$\left(\vec{\nabla} \times \vec{F}\right)\left(\vec{R}(\theta,\phi)\right) = (2yz, -2xz, -x^2 - y^2) = \left(8\sin\phi\cos\phi\sin\theta, -8\sin\phi\cos\phi\cos\theta, -4\sin^2\phi\right)$$

$$\iint_S \vec{\nabla} \times \vec{F} \bullet d\vec{S} = \iint_S \vec{\nabla} \times \vec{F} \bullet \vec{N}\ d\theta\ d\phi$$

$$= \int\int \left(32\sin^3\phi\cos\phi\sin\theta\cos\theta - 32\sin^3\phi\cos\phi\sin\theta\cos\theta - 16\sin^3\phi\cos\phi\right)\ d\theta\ d\phi$$

$$= \int_0^{\pi/2}\int_0^{2\pi} \left(-16\sin^3\phi\cos\phi\right)\ d\theta\ d\phi = 2\pi(-16)\left[\frac{\sin^4\phi}{4}\right]_0^{\pi/2} = -8\pi$$

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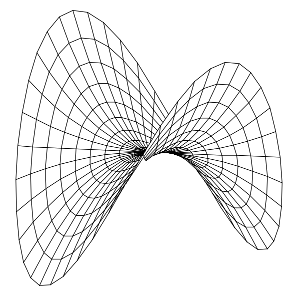
9. (20 points)

The spider web at the right is the graph of the hyperbolic paraboloid $z = xy$.

It may be parametrized as

$\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, r^2\sin\theta\cos\theta)$.

Find the area of the web for $r \le \sqrt{8}$.



$$\vec{R}_r = (\cos\theta,\quad \sin\theta,\quad 2r\sin\theta\cos\theta)$$

$$\vec{R}_\theta = \left(-r\sin\theta,\quad r\cos\theta,\quad r^2\cos^2\theta - r^2\sin^2\theta\right)$$

$$\vec{N} = i\left(r^2\cos^2\theta\sin\theta - r^2\sin^3\theta - 2r^2\sin\theta\cos^2\theta\right) - j\left(r^2\cos^3\theta - r^2\sin^2\theta\cos\theta + 2r^2\sin^2\theta\cos\theta\right)$$
$$+ k\left(r\cos^2\theta + r\sin^2\theta\right)$$
$$= \left(-r^2\sin^3\theta - r^2\sin\theta\cos^2\theta, -r^2\cos^3\theta - r^2\sin^2\theta\cos\theta, r\right) = (-r^2\sin\theta, -r^2\cos\theta, r)$$

$$\left|\vec{N}\right| = \sqrt{r^4\sin^2\theta + r^4\cos^2\theta + r^2} = \sqrt{r^4 + r^2} = r\sqrt{r^2+1}$$

$$A = \iint 1\ dS = \iint\left|\vec{N}\right|\ dr\ d\theta = \int_0^{2\pi}\int_0^{\sqrt{8}} r\sqrt{r^2+1}\ dr\ d\theta = 2\pi\left[\frac{(r^2+1)^{3/2}}{3}\right]_0^{\sqrt{8}}$$

$$= 2\pi\left[\frac{(9)^{3/2}}{3}\right] - 2\pi\left[\frac{(1)^{3/2}}{3}\right] = 2\pi\left[\frac{27}{3}\right] - 2\pi\left[\frac{1}{3}\right] = \frac{52\pi}{3}$$