MATH 253 Honors FINAL EXAM Spring 1999

Sections 201-203 Solutions P. Yasskin

1. Find the volume of the parallelepiped with edges $(3,2,0)$, $(-1,1,2)$ and $(0,4,1)$.

- **a.** -23
- **b.** -19
- **c.** 19 correctchoice
- **d.** 21
- **e.** 23

$$\begin{vmatrix} 3 & 2 & 0 \\ -1 & 1 & 2 \\ 0 & 4 & 1 \end{vmatrix} = 3\begin{vmatrix} 1 & 2 \\ 4 & 1 \end{vmatrix} - 2\begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = 3(1-8) - 2(-1) = -19$$

Volume must be positive: $V = 19$

2. Find the unit tangent vector $\hat{T}$ to the curve $\vec{r}(t) = (3t, 2t^2, 4t^3)$ at the point $\vec{r}(1) = (3,2,4)$.

- **a.** $\left(\frac{3}{13}, \frac{4}{13}, \frac{12}{13}\right)$ correctchoice
- **b.** $\left(\frac{3}{\sqrt{29}}, \frac{4}{\sqrt{29}}, \frac{12}{\sqrt{29}}\right)$
- **c.** $\left(\frac{3}{169}, \frac{4}{169}, \frac{12}{169}\right)$
- **d.** $\left(\frac{3}{29}, \frac{4}{29}, \frac{12}{29}\right)$
- **e.** $\left(\frac{3}{169}, \frac{-4}{169}, \frac{12}{169}\right)$

$\vec{v} = (3, 4t, 12t^2)$ $\vec{v}(1) = (3,4,12)$ $|\vec{v}| = \sqrt{9+16+144} = 13$ $\hat{T} = \frac{\vec{v}}{|\vec{v}|} = \left(\frac{3}{13}, \frac{4}{13}, \frac{12}{13}\right)$

3. If a jet flies around the world from East to West, directly above the equator, in what direction does the unit binormal $\hat{B}$ point?

a. North
b. South correctchoice
c. East
d. West
e. Down (toward the center of the earth)

$\hat{T}$ points West, $\hat{N}$ points Down and by the right hand rule $\hat{B}$ points South.

4. At the point (x,y,z) where the line $\vec{r}(t) = (2+t, 3-t, t)$ intersects the plane $2x - y + z = 5,$ we have $x + y + z =$

a. 2
b. 3
c. 4
d. 5
e. 6 correctchoice

Plug the line into the plane and solve for t:
$2(2+t) - (3-t) + (t) = 5 \qquad 4t + 1 = 5 \qquad t = 1$
Plug back into the line:
$(x,y,z) = (2+t, 3-t, t) = (3,2,1)$ So $x + y + z = 6$

5. The temperature in an ideal gas is given by $T = \kappa\frac{P}{\rho}$ where κ is a constant, P is the pressure and ρ is the density. At a certain point $Q = (1,2,3),$ we have

$$P(Q) = 4 \qquad \vec{\nabla}P(Q) = (-3,2,1)$$
$$\rho(Q) = 2 \qquad \vec{\nabla}\rho(Q) = (3,-1,2)$$

So at the point Q, the temperature is $T(Q) = 2\kappa$ and its gradient is $\vec{\nabla}T(Q) =$

a. $\kappa(-4.5, 0, 2.5)$
b. $\kappa(1.5, 0, 2.5)$
c. $\kappa(1.5, 2, -4.5)$
d. $\kappa(-4.5, 2, -1.5)$ correctchoice
e. $\kappa(-1.5, 2, 2.5)$

By chain rule: (Think about each component separately.)

$$\vec{\nabla}T = \frac{\partial T}{\partial P}\vec{\nabla}P + \frac{\partial T}{\partial \rho}\vec{\nabla}\rho = \frac{\kappa}{\rho}\vec{\nabla}P - \frac{\kappa P}{\rho^2}\vec{\nabla}\rho = \frac{\kappa}{2}(-3,2,1) - \frac{\kappa 4}{2^2}(3,-1,2)$$
$$= \kappa\left(-\frac{3}{2}, 1, \frac{1}{2}\right) + \kappa(-3,1,-2) = \kappa\left(-\frac{9}{2}, 2, -\frac{3}{2}\right)$$

6. The saddle surface $z = xy$ may be parametrized as $\vec{R}(u,v) = (u,v,uv)$. Find the plane tangent to the surface at the point $(1,2,2)$.

a. $3x + y - z = 3$

b. $2x + y - z = 2$ correctchoice

c. $3x + 2y - z = 5$

d. $2x - y + z = 2$

e. $3x - y + z = 3$

METHOD I: Plane tangent to a graph: $z = f(x,y) = xy$

$$f = xy \qquad f_x = y \qquad f_y = x$$

$$f(1,2) = 2 \qquad f_x(1,2) = 2 \qquad f_y(1,2) = 1$$

$$z = f(1,2) + f_x(1,2)(x-1) + f_y(1,2)(y-2) = 2 + 2(x-1) + 1(y-2) = 2x + y - 2$$

METHOD II: Plane tangent to a parametric surface: $\vec{R}(u,v) = (u,v,uv)$

$\vec{R}_u = (1,0,v)$ $\quad \vec{R}_v = (0,1,u)$ $\quad \vec{N} = \vec{R}_u \times \vec{R}_v = (-v,-u,1)$

At $P = (1,2,2)$ $\quad u = 1, v = 2$ $\quad$ So $\vec{N} = (-2,-1,1)$.

$\vec{N} \bullet \vec{X} = \vec{N} \bullet P$ $\qquad -2x - y + z = -2 - 2 + 2 = -2$

7. Find the minimum value of the function $f = x^2 + y^2 + z^2$ on the plane $x + 2y + 3z = 14$.

a. 0

b. $\frac{7}{4}$

c. $\frac{7}{2}$

d. 14 correctchoice

e. 28

METHOD I: Lagrange Multipliers

$\vec{\nabla} f = (2x,2y,2z)$ $\quad \vec{\nabla} g = (1,2,3)$ $\quad \vec{\nabla} f = \lambda \vec{\nabla} g$

$$2x = \lambda \quad 2y = 2\lambda \quad 2z = 3\lambda$$

$$x = \frac{\lambda}{2} \quad y = \lambda \quad z = \frac{3\lambda}{2}$$

Use the constraint: $\frac{\lambda}{2} + 2\lambda + 3\left(\frac{3\lambda}{2}\right) = 14$ $\quad 7\lambda = 14$ $\quad \lambda = 2$

$x = 1$ $\quad y = 2$ $\quad z = 3$ $\qquad f(1,2,3) = 1 + 4 + 9 = 14$

METHOD II: Eliminate a Variable:

$x = 14 - 2y - 3z$ $\qquad f = (14 - 2y - 3z)^2 + y^2 + z^2$

$$\vec{\nabla} f = \left(\frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = (-4(14 - 2y - 3z) + 2y, -6(14 - 2y - 3z) + 2z) = (0,0)$$

$10y + 12z = 56$ $\quad 12y + 20z = 84$ $\quad \Rightarrow \quad y = 2, z = 3$

So $x = 1$ and $f(1,2,3) = 1 + 4 + 9 = 14$

8. Compute $\int_0^3 \int_{y^2}^9 y\cos(x^2)\ dx\ dy$

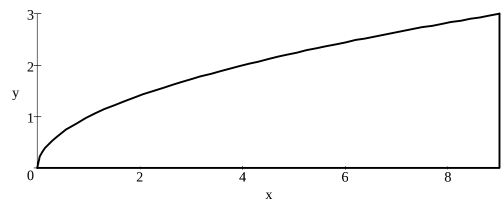
a. $\frac{1}{4}\sin 81$ correctchoice

b. $\frac{1}{2}\cos 9 - \frac{1}{2}$

c. $\frac{9}{2}\sin 81 + \cos 9 - 1$

d. $-\frac{9}{2}\sin 81 + \frac{9}{2}\sin y^4$

e. $\frac{9}{2}\sin 81 - \cos 9 + 1$



Reverse the order of integration:

$$\int_0^3 \int_{y^2}^9 y\cos(x^2)\ dx\ dy = \int_0^9 \int_0^{\sqrt{x}} y\cos(x^2)\ dy\ dx = \int_0^9 \left[\frac{y^2}{2}\cos(x^2) \right]_{y=0}^{\sqrt{x}} dx$$

$$= \int_0^9 \frac{x}{2}\cos(x^2)\ dx = \frac{1}{4}\sin(x^2)\Big|_{x=0}^{9} = \frac{1}{4}\sin 81$$

9. Compute $\iiint z^2\ dV$ over the solid sphere $x^2 + y^2 + z^2 \le 4$.

a. $\frac{64\pi}{5}$

b. $\frac{256\pi}{3}$

c. $\frac{48\pi}{5}$

d. $\frac{64\pi}{15}$

e. $\frac{128\pi}{15}$ correctchoice

In spherical coordinates:

$$x = \rho\sin\phi\cos\theta$$
$$y = \rho\sin\phi\sin\theta \qquad J = \rho^2\sin\phi$$
$$z = \rho\cos\phi$$

$$\iiint z^2\ dV = \int_0^{2\pi}\int_0^{\pi}\int_0^2 \rho^2\cos^2\phi\ \rho^2\sin\phi\ d\rho\ d\phi\ d\theta = 2\pi\left[-\frac{\cos^3\phi}{3} \right]_{\phi=0}^{\pi} \left[\frac{\rho^5}{5} \right]_{\rho=0}^{2}$$

$$= 2\pi\left[-\frac{-1}{3} - -\frac{1}{3} \right]\left[\frac{32}{5} \right] = \frac{128\pi}{15}$$

10. Compute $\iint \vec{F} \bullet d\vec{S}$ for $\vec{F} = (x, y^3, z)$ over the surface of the cube $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$ with outward normal.

- **a**. 1
- **b**. 2
- **c**. 3 correctchoice
- **d**. 4
- **e**. 6

$\vec{\nabla} \bullet \vec{F} = 1 + 3y^2 + 1 = 2 + 3y^2$

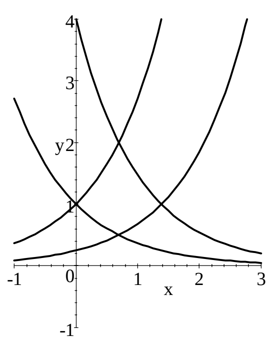
By Gauss' Theorem:

$$\iint \vec{F} \bullet d\vec{S} = \iiint \vec{\nabla} \bullet \vec{F}\ dV = \int_0^1 \int_0^1 \int_0^1 2 + 3y^2\ dx\ dy\ dz = \int_0^1 1\ dx \int_0^1 1\ dz \int_0^1 2 + 3y^2\ dy$$
$$= [2y + y^3]_0^1 = 3$$

11. (15 points) Find the area of the diamond shaped region between the curves

$$y = e^x, \quad y = \frac{1}{4}e^x, \quad y = e^{-x} \quad \text{and} \quad y = 4e^{-x}.$$

You **must** use the curvilinear coordinates $u = ye^{-x}$ and $v = ye^x$.



$$\frac{1}{4} \le u \le 1 \qquad 1 \le v \le 4$$
$$\frac{v}{u} = \frac{ye^x}{ye^{-x}} = e^{2x} \qquad x = \frac{1}{2}\ln\left(\frac{v}{u}\right) = \frac{1}{2}\ln(v) - \frac{1}{2}\ln(u)$$
$$uv = y^2 \qquad y = \sqrt{uv} = u^{1/2}v^{1/2}$$

$$\frac{\partial x}{\partial u} = -\frac{1}{2u} \qquad \frac{\partial x}{\partial v} = \frac{1}{2v}$$
$$\frac{\partial y}{\partial u} = \frac{v^{1/2}}{2u^{1/2}} \qquad \frac{\partial y}{\partial v} = \frac{u^{1/2}}{2v^{1/2}}$$

$$J = \left| \begin{vmatrix} -\frac{1}{2u} & \frac{1}{2v} \\ \frac{v^{1/2}}{2u^{1/2}} & \frac{u^{1/2}}{2v^{1/2}} \end{vmatrix} \right| = \left| -\frac{1}{2u}\frac{u^{1/2}}{2v^{1/2}} - \frac{1}{2v}\frac{v^{1/2}}{2u^{1/2}} \right| = \left| -\frac{1}{4u^{1/2}v^{1/2}} - \frac{1}{4v^{1/2}u^{1/2}} \right| = \frac{1}{2u^{1/2}v^{1/2}}$$

$$A = \iint 1\ dA = \int_1^4 \int_{1/4}^1 J\ du\ dv = \int_1^4 \int_{1/4}^1 \frac{1}{2}u^{-1/2}v^{-1/2}\ du\ dv = \frac{1}{2}\int_1^4 v^{-1/2}\ dv \int_{1/4}^1 u^{-1/2}\ du$$
$$= \frac{1}{2}\left[2v^{1/2}\right]_1^4 \left[2u^{1/2}\right]_{1/4}^1 = 2[2-1]\left[1 - \frac{1}{2}\right] = 2 \bullet 1 \bullet \frac{1}{2} = 1$$

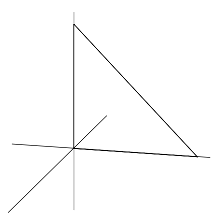
12. (10 points) Find the mass of a wire in the shape of the curve $y = \ln(\cos x)$ for $0 \le x \le \frac{\pi}{4}$ if the density is $\rho = \frac{\sin x}{e^y}$.
Note: The wire may be parametrized as $\vec{r}(t) = (t, \ln(\cos t))$.

$\vec{v} = \left(1, \frac{-\sin t}{\cos t}\right) = (1, -\tan t)$ $\quad |\vec{v}| = \sqrt{1+\tan^2 t} = \sec t$ for $0 \le t \le \frac{\pi}{4}$

$\rho = \frac{\sin x}{e^y} = \frac{\sin t}{e^{\ln(\cos t)}} = \frac{\sin t}{\cos t} = \tan t$

$M = \int \rho \; ds = \int \rho(t) \; |\vec{v}| \; dt = \int_0^{\pi/4} \tan t \; \sec t \; dt = \sec t \Big|_0^{\pi/4} = \sqrt{2} - 1$

13. (10 points) Compute $\oint x \; dx + z \; dy - y \; dz$ around the boundary of the triangle with vertices $(0,0,0)$, $(0,1,0)$ and $(0,0,1)$, traversed in this order of the vertices. Hint: The yz-plane may be parametrized as $\vec{R}(u,v) = (0,u,v)$.



Let $\vec{F} = (x, z, -y)$.

By Stokes' Theorem, the integral is

$\oint \vec{F} \bullet d\vec{s} = \iint \vec{\nabla} \times \vec{F} \bullet d\vec{S}$.

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ x & z & -y \end{vmatrix} = (-2, 0, 0)$$

Using the parametrization of the triangle

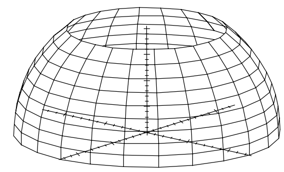
$\vec{R}_u = (0,1,0) = \hat{\jmath}$ $\quad \vec{R}_v = (0,0,1) = \hat{k}$ $\quad \vec{N} = \hat{\jmath} \times \hat{k} = \hat{\imath} = (1,0,0)$

Since the triangle is traversed counterclockwise as seen from the positive x-axis, the normal is in the correct direction. So

$$\oint \vec{F} \bullet d\vec{s} = \iint \vec{\nabla} \times \vec{F} \bullet \vec{N} \; du \; dv = \int_0^1 \int_0^{1-u} (-2) \; dv \; du = -2 \int_0^1 \Big[v \Big]_{v=0}^{1-u} du$$

$$= -2 \int_0^1 (1-u) \; du = -2 \left[u - \frac{u^2}{2} \right]_0^1 = -2 \left[1 - \frac{1}{2} \right] = -1$$

14. (15 points) Compute $\displaystyle\iint_S \vec{\nabla} \times \vec{F} \bullet d\vec{S}$



for $\vec{F} = (x^2y, y, z^2)$ over the piece of the sphere $x^2 + y^2 + z^2 = 25$ for $0 \le z \le 4$ with normal pointing away from the z-axis.

Hint: Parametrize the upper and lower edges.

By Stokes' Theorem $\displaystyle\iint_S \vec{\nabla} \times \vec{F} \bullet d\vec{S} = \oint_{\partial S} \vec{F} \bullet d\vec{s} = \oint_{\text{upper}} \vec{F} \bullet d\vec{s} + \oint_{\text{lower}} \vec{F} \bullet d\vec{s}$

By the right hand rule, since the normal points outward, the **upper** circle must be traversed **clockwise** while the **lower** circle must be traversed **counterclockwise** as seen from the positive z-axis. We compute each line integral:

Upper Circle: $z = 4$ $\quad x^2 + y^2 = 25 - z^2 = 25 - 16 = 9$

$\vec{r}(t) = (3\cos t, 3\sin t, 4)$ $\quad \vec{v} = (-3\sin t, 3\cos t, 0)$

This is clockwise, so we reverse the velocity: $\vec{v} = (3\sin t, -3\cos t, 0)$

$\vec{F} = (x^2y, y, z^2) = (27\cos^2 t\sin t, 3\sin t, 16)$

$$\oint_{\text{upper}} \vec{F} \bullet d\vec{s} = \oint_{\text{upper}} \vec{F} \bullet \vec{v}\, dt = \int_0^{2\pi} \left(81\cos^2 t\sin^2 t - 9\sin t\cos t\right)\, dt$$

$$= \int_0^{2\pi} \left(\frac{81}{4}\sin^2(2t) - \frac{9}{2}\sin(2t)\right)\, dt = \frac{81}{4}\frac{1}{2}(2\pi) + \frac{9}{2}\frac{\cos(2t)}{2} = \frac{81\pi}{4}$$

Lower Circle: $z = 0$ $\quad x^2 + y^2 = 25 - z^2 = 25$

$\vec{r}(t) = (5\cos t, 5\sin t, 0)$ $\quad \vec{v} = (-5\sin t, 5\cos t, 0)$

This is counterclockwise, so we do not need to reverse the velocity.

$\vec{F} = (x^2y, y, z^2) = (125\cos^2 t\sin t, 5\sin t, 0)$

$$\oint_{\text{lower}} \vec{F} \bullet d\vec{s} = \oint_{\text{lower}} \vec{F} \bullet \vec{v}\, dt = \int_0^{2\pi} \left(-625\cos^2 t\sin^2 t + 25\sin t\cos t\right)\, dt$$

$$= \int_0^{2\pi} \left(-\frac{625}{4}\sin^2(2t) + \frac{25}{2}\sin(2t)\right)\, dt = -\frac{625}{4}\frac{1}{2}(2\pi) - \frac{25}{2}\frac{\cos(2t)}{2} = -\frac{625\pi}{4}$$

Total Boundary:

$$\iint_S \vec{\nabla} \times \vec{F} \bullet d\vec{S} = \frac{81\pi}{4} - \frac{625\pi}{4} = -\frac{544\pi}{4} = -136\pi$$