MATH 253 Honors EXAM 1 Fall 1999

Sections 201-202 Solutions P. Yasskin

Multiple Choice: (8 points each)

**1**. Find the volume of the parallelepiped with edges $\vec{a} = (2,4,0)$, $\vec{b} = (-1,0,3)$ and $\vec{c} = (0,1,-2)$.

**a**. $-14$

**b**. $-12$

**c**. $2$

**d**. $12$

**e**. $14$ correctchoice

$$\vec{a} \bullet \left(\vec{b} \times \vec{c}\right) = \begin{vmatrix} 2 & 4 & 0 \\ -1 & 0 & 3 \\ 0 & 1 & -2 \end{vmatrix} = -14 \qquad V = \left|\vec{a} \bullet \left(\vec{b} \times \vec{c}\right)\right| = |-14| = 14$$

**2**. At the point where the line $\dfrac{x+4}{3} = \dfrac{y-1}{2} = \dfrac{z+2}{4}$ intersects the plane $(x,y,z) = (4-2s+t, 3+2s, 6-t)$ we have $2x+y+z =$

**a**. $15$ correctchoice

**b**. $13$

**c**. $11$

**d**. $9$

**e**. $7$

Substitute the plane $x = 4-2s+t, \quad y = 3+2s, \quad z = 6-t$ into the line:

$$\frac{4-2s+t+4}{3} = \frac{3+2s-1}{2} = \frac{6-t+2}{4} \qquad \Rightarrow \qquad \frac{8-2s+t}{3} = 1+s = 2-\frac{t}{4}$$

From the second and third parts: $s = 1 - \dfrac{t}{4}$

From the first and third parts: $\dfrac{8-2\left(1-\frac{t}{4}\right)+t}{3} = 2 - \dfrac{t}{4}$

Solve for $t$: Multiply by $12$: $4\left(8-2\left(1-\frac{t}{4}\right)+t\right) = 3(8-t)$

$$32-2(4-t)+4t = 24-3t \qquad \Rightarrow \qquad 24+6t = 24-3t \qquad \Rightarrow \qquad 9t = 0 \qquad \Rightarrow \qquad t = 0$$

Substitute back: $s = 1, \quad x = 4-2 = 2, \quad y = 3+2 = 5, \quad z = 6,$

$2x+y+z = 15$

**3**. Find the equation of the plane which is perpendicular to the line through $P = (1,3,5)$ and $Q = (3,7,1)$ and is equidistant from $P$ and $Q$.

- **a**. $x + 2y - 2z = 12$
- **b**. $x + 2y - 2z = 6$ correctchoice
- **c**. $x + 2y - 2z = -6$
- **d**. $x + 2y - 2z = -3$
- **e**. $2x + 4y - 4z = 24$

The plane passes through the midpoint $R = \dfrac{P+Q}{2} = (2,5,3)$.

Its normal is $\vec{N} = \overrightarrow{PQ} = Q - P = (2,4,-4)$.

So its equation is $\vec{N} \bullet X = \vec{N} \bullet R$ or $2x + 4y - 4z = 2 \bullet 2 + 4 \bullet 5 - 4 \bullet 3 = 12$ or $x + 2y - 2z = 6$

---

**Problems 4 – 6**: The temperature in an ideal gas is given by $T = \kappa \dfrac{P}{\rho}$ where $\kappa$ is a constant, $P$ is the pressure and $\rho$ is the density. The pressure, density and temperature are all functions of position. At a certain point $Q = (1,2,3)$, we have

$$P(Q) = 4 \qquad \vec{\nabla}P(Q) = (-3,2,1)$$

$$\rho(Q) = 2 \qquad \vec{\nabla}\rho(Q) = (3,-1,2)$$

**4**. Use the linear approximation to estimate the pressure at the point $R = (1.2, 1.9, 3.1)$.

- **a**. 2.9
- **b**. 3.1
- **c**. 3.3 correctchoice
- **d**. 4.7
- **e**. 7.3

The linear approximation says:

$$P(x,y,z) \approx P(a,b,c) + \frac{\partial P}{\partial x}(x-a) + \frac{\partial P}{\partial y}(y-b) + \frac{\partial P}{\partial z}(z-c)$$

Here $(x,y,z) = R = (1.2,1.9,3.1)$, $(a,b,c) = Q = (1,2,3)$,

and $\left(\dfrac{\partial P}{\partial x}, \dfrac{\partial P}{\partial y}, \dfrac{\partial P}{\partial z}\right) = \vec{\nabla}P(Q) = (-3,2,1)$

So $P(R) \approx 4 - 3(1.2-1) + 2(1.9-2) + 1(3.1-3) = 4 - 3(.2) + 2(-.1) + 1(.1) = 3.3$

**5**. At the point $Q$, the temperature is $T(Q) = 2\kappa$ and its gradient is $\vec{\nabla} T(Q) =$

**a**. $\kappa(-4.5, 0, 2.5)$
**b**. $\kappa(1.5, 0, 2.5)$
**c**. $\kappa(1.5, 2, -4.5)$
**d**. $\kappa(-4.5, 2, -1.5)$ correctchoice
**e**. $\kappa(-1.5, 2, 2.5)$

Apply chain rule with $T = \kappa\frac{P}{\rho}$ and $P = P(x,y,z)$ and $\rho = \rho(x,y,z)$:

$\frac{\partial T}{\partial x} = \frac{\partial T}{\partial P}\frac{\partial P}{\partial x} + \frac{\partial T}{\partial \rho}\frac{\partial \rho}{\partial x} = \frac{\kappa}{\rho}\frac{\partial P}{\partial x} - \frac{\kappa P}{\rho^2}\frac{\partial \rho}{\partial x}$ and similarly for $\frac{\partial T}{\partial y}$ and $\frac{\partial T}{\partial y}$

These fit together as the vector gradient:

$$\vec{\nabla} T = \frac{\kappa}{\rho}\vec{\nabla} P - \frac{\kappa P}{\rho^2}\vec{\nabla}\rho = \frac{\kappa}{2}(-3,2,1) - \frac{\kappa 4}{2^2}(3,-1,2) = \kappa\left(-\frac{3}{2} - 3, 1+1, \frac{1}{2} - 2\right)$$
$$= \kappa(-4.5, 2, -1.5)$$

**6**. If a fly is located at the point $Q$ and travelling with velocity $\vec{v} = (3,4,12)$, how fast is the density changing at the location of the fly?

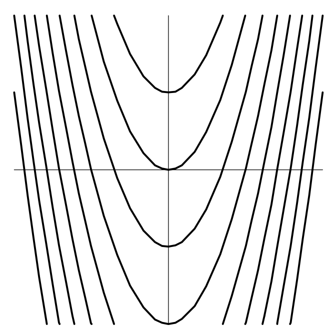
**a**. $\frac{d\rho}{dt}(Q) = 37$
**b**. $\frac{d\rho}{dt}(Q) = 29$ correctchoice
**c**. $\frac{d\rho}{dt}(Q) = 21$
**d**. $\frac{d\rho}{dt}(Q) = 2.1$
**e**. $\frac{d\rho}{dt}(Q) = .21$

Chain rule:

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial x}\frac{dx}{dt} + \frac{\partial \rho}{\partial y}\frac{dy}{dt} + \frac{\partial \rho}{\partial z}\frac{dz}{dt} = \vec{v} \bullet \vec{\nabla}\rho = (3,4,12) \bullet (3,-1,2) = 9 - 4 + 24 = 29$$

**7**. The graph at the right is the contour plot of which function?

a. $y^2 - x^2$
b. $xy$
c. $x^2 + y^2$
d. $y - x^2$ correctchoice
e. $x - y^2$



The contour plot is the level curves $y - x^2 = C$ for various values of $C$. These are parabolas opening up.

**8**. (12 points) Does each limit exist? Why or why not? Find the value of the one that exists. (***Up to* 4 *points extra credit for a good explanation***.)

**a**. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^4+y^2}$

First check the straight lines $y = mx$:

$$\lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^4+y^2} = \lim_{\substack{x\to 0\\ y=mx}} \frac{2x^2mx}{x^4+m^2x^2} = \lim_{x\to 0} \frac{2mx}{x^2+m^2} = 0$$

So all straight lines give $0$. Now check the parabola $y = x^2$:

$$\lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^4+y^2} = \lim_{\substack{x\to 0\\ y=x^2}} \frac{2x^2x^2}{x^4+x^4} = \lim_{x\to 0} \frac{2}{2} = 1$$

This is different. So the limit does not exist.

**b**. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^2+y^2}$

First check the straight lines $y = mx$:

$$\lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^2+y^2} = \lim_{\substack{x\to 0\\ y=mx}} \frac{2x^2mx}{x^2+m^2x^2} = \lim_{x\to 0} \frac{2mx}{1+m^2} = 0$$

Now check the parabola $y = mx^2$

$$\lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^2+y^2} = \lim_{\substack{x\to 0\\ y=mx^2}} \frac{2x^2mx^2}{x^2+m^2x^4} = \lim_{x\to 0} \frac{2mx^2}{1+m^2x^2} = 0$$

So try polar: $x = r\cos\theta$ $y = r\sin\theta$

$$\lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^2+y^2} = \lim_{r\to 0} \frac{2r^3\cos^2\theta\sin\theta}{r^2} = \lim_{r\to 0} 2r\cos^2\theta\sin\theta = 0$$

since $\cos^2\theta\sin\theta$ is bounded and $r \to 0$.

**9**. (16 points) Consider the parametric curve $\vec{r}(t) = (t, 4e^{t/2}, 2e^t)$.

**a.** Compute the velocity and acceleration:

$\vec{v} = (1, 2e^{t/2}, 2e^t)$

$\vec{a} = (0, e^{t/2}, 2e^t)$

**b.** Find a parametric equation for the line tangent to the curve at $t = 0$.

$\vec{r}(0) = (0,4,2) \qquad \vec{v}(0) = (1,2,2)$
$X = (x,y,z) = \vec{r}(0) + t\vec{v}(0) = (0,4,2) + t(1,2,2) = (t, 4+2t, 2+2t)$
or $\quad x = t, \quad y = 4+2t, \quad z = 2+2t$

**c.** Find a parametric equation for the plane instantaneously containing the curve at $t = 0$. This is the plane containing the velocity and the acceleration.

$\vec{r}(0) = (0,4,2) \qquad \vec{v}(0) = (1,2,2) \qquad \vec{a}(0) = (0,1,2)$
$X = (x,y,z) = \vec{r}(0) + t\vec{v}(0) + s\vec{a}(0) = (0,4,2) + t(1,2,2) + s(0,1,2)$
$\quad = (t, 4+2t+s, 2+2t+2s)$
or $\quad x = t, \quad y = 4+2t+s, \quad z = 2+2t+2s$

**d.** Find a non-parametric equation for the plane instantaneously containing the curve at $t = 0$.

The normal to the plane is

$$\vec{N} = \vec{v}(0) \times \vec{a}(0) = (1,2,2) \times (0,1,2) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 2 & 2 \\ 0 & 1 & 2 \end{vmatrix} = i(4-2) - j(2) + k(1) = (2,-2,1)$$

The point is $\vec{r}(0) = (0,4,2)$. So the plane is $\vec{N} \bullet X = \vec{N} \bullet \vec{r}(0)$ or
$2x - 2y + z = 2 \bullet 0 - 2 \bullet 4 + 2$ or $2x - 2y + z = -6$.

**e.** Find the arclength of the curve between $t = 0$ and $t = 2$.
HINT: Factor inside the square root.
$|\vec{v}| = \sqrt{1 + 4e^t + 4e^{2t}} = \sqrt{(1+2e^t)^2} = 1 + 2e^t$
$L = \int_0^2 |\vec{v}|\,dt = \int_0^2 1 + 2e^t\,dt = \left[t + 2e^t\right]_0^2 = [2 + 2e^2] - [0 + 2] = 2e^2$

**10**. (16 points) Find the equation of the plane tangent to the graph of the function $f(x,y) = 3xe^y - 2x^2e^{2y}$ at the point $(x,y) = (1, \ln 2)$.

$f(x,y) = 3xe^y - 2x^2e^{2y} \qquad f(1,\ln 2) = 3 \bullet 2 - 2 \bullet 4 = -2$
$f_x(x,y) = 3e^y - 4xe^{2y} \qquad f_x(1,\ln 2) = 3 \bullet 2 - 4 \bullet 4 = -10$
$f_y(x,y) = 3xe^y - 4x^2e^{2y} \qquad f_y(1,\ln 2) = 3 \bullet 2 - 4 \bullet 4 = -10$
$z = f(1,\ln 2) + f_x(1,\ln 2)(x-1) + f_y(1,\ln 2)(y - \ln 2) = -2 - 10(x-1) - 10(y - \ln 2)$
$z = -10x - 10y + 8 + 10\ln 2$