MATH 253 Honors EXAM 2 Fall 1999

Sections 201-202 Solutions P. Yasskin

Multiple Choice: (10 points each) Work Out: (15 points each)

1. Compute $\int_0^2 \int_y^2 (x+y)\,dx\,dy$.

a. 2
b. 4 correctchoice
c. 6
d. 8
e. 10

$$\int_0^2 \int_y^2 (x+y)\,dx\,dy = \int_0^2 \left[\frac{x^2}{2} + xy \right]_{x=y}^2 dy = \int_0^2 [2+2y] - \left[\frac{y^2}{2} + y^2 \right] dy = \int_0^2 \left[2 + 2y - \frac{3y^2}{2} \right] dy$$
$$= \left[2y + y^2 - \frac{y^3}{2} \right]_0^2 = 4 + 4 - 4 = 4$$

2. Compute $\iiint_R 2xy\,dV$ over the solid region R given by $x^2 \le y \le x$ and $0 \le z \le x$.

a. $\frac{1}{70}$
b. $\frac{1}{35}$
c. $\frac{2}{35}$ correctchoice
d. $\frac{4}{35}$
e. None of these.

$$\iiint_R 2xy\,dV = \int_0^1 \int_{x^2}^x \int_0^x 2xy\,dz\,dy\,dx = \int_0^1 \int_{x^2}^x \left[2xyz \right]_{z=0}^x dy\,dx = \int_0^1 \int_{x^2}^x 2x^2y\,dy\,dx = \int_0^1 \left[x^2y^2 \right]_{y=x^2}^x dx$$
$$= \int_0^1 (x^4 - x^6)\,dx = \left[\frac{x^5}{5} - \frac{x^7}{7} \right]_{x=0}^1 = \left[\frac{1}{5} - \frac{1}{7} \right] = \frac{2}{35}$$

3. Compute $\iint_R e^{x^2+y^2}\,dA$ over the region R in the 1^{st} quadrant between the circles $x^2+y^2 = 4$ and $x^2+y^2 = 9$.

a. $\frac{\pi}{2}e^5$
b. $\frac{\pi}{4}(e^3 - e^2)$
c. $\frac{\pi}{2}(e^3 - e^2)$
d. $\frac{\pi}{4}(e^9 - e^4)$ correctchoice
e. $\frac{\pi}{2}(e^9 - e^4)$

$$\iint_R e^{x^2+y^2}\,dA = \int_0^{\pi/2} \int_2^3 e^{r^2} r\,dr\,d\theta = \frac{\pi}{2} \left[\frac{1}{2} e^{r^2} \right]_{r=2}^3 = \frac{\pi}{4}(e^9 - e^4)$$

4. Compute $\int_0^2 \int_y^2 e^{-x^2}\,dx\,dy$.

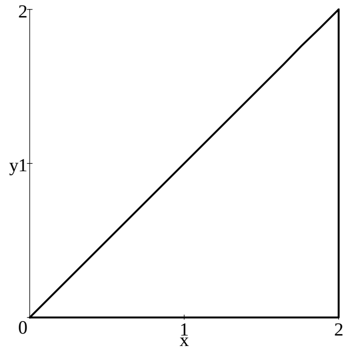
a. $\frac{1}{2}(1-e^{-4})$ correctchoice

b. $\frac{1}{4}(1-e^{-4})$

c. $\frac{1}{4}(e^{-4}-1)$

d. $\frac{1}{4}e^{-4}$

e. $-\frac{1}{2}e^{-4}$



$$\int_0^2 \int_y^2 e^{-x^2}\,dx\,dy = \int_0^2 \int_0^x e^{-x^2}\,dy\,dx = \int_0^2 \left[e^{-x^2}y\right]_{y=0}^{x} dx = \int_0^2 e^{-x^2}x\,dx \qquad u = -x^2 \qquad du - -2xdx$$

$$= -\frac{1}{2}\int e^u\,du = -\frac{1}{2}e^u = -\frac{1}{2}\left[e^{-x^2}\right]_{x=0}^{2} = -\frac{1}{2}(e^{-4}-1) = \frac{1}{2}(1-e^{-4})$$

5. A cupcake has its base on the xy-plane. Its sides are the cylinder $x^2+y^2=4$ and its top is the paraboloid $z=6-x^2-y^2$. Its density is $\rho = 3\,\frac{\text{gm}}{\text{cm}^3}$. Find its total mass and the z-component of its center of mass.

$$M = \iiint \rho\,dV = \int_0^{2\pi}\int_0^2\int_0^{6-r^2} 3r\,dz\,dr\,d\theta = 2\pi\int_0^2 \left[3rz\right]_{z=0}^{6-r^2} dr = 6\pi\int_0^2 r(6-r^2)\,dr$$

$$= 6\pi\left[6\frac{r^2}{2}-\frac{r^4}{4}\right]_0^2 = 6\pi(12-4) = 48\pi$$

$$z\text{-mom} = \iiint z\rho\,dV = \int_0^{2\pi}\int_0^2\int_0^{6-r^2} z3r\,dz\,dr\,d\theta = 2\pi\int_0^2\left[3r\frac{z^2}{2}\right]_{z=0}^{6-r^2} dr = 3\pi\int_0^2 r(6-r^2)^2\,dr$$

$$= 3\pi\left[\frac{(6-r^2)^3}{-6}\right]_0^2 = -\frac{\pi}{2}(8-216) = 104\pi$$

$$\bar{z} = \frac{z\text{-mom}}{M} = \frac{104\pi}{48\pi} = \frac{13}{6}$$

6. Find the mass and the z-component of the center of mass of the hemisphere $0 \le z \le \sqrt{25-x^2-y^2}$ whose density is given by $\delta = \frac{1}{5}(x^2+y^2+z^2)$.

$$M = \iiint \delta\,dV = \int_0^{\pi/2}\int_0^{2\pi}\int_0^5 \frac{1}{5}\rho^2 \bullet \rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi = \left[-\cos\varphi\right]_0^{\pi/2}[2\pi]\left[\frac{\rho^5}{25}\right]_0^5$$

$$= [1][2\pi][125] = 250\pi$$

$$z\text{-mom} = \iiint z\delta\,dV = \int_0^{\pi/2}\int_0^{2\pi}\int_0^5 \rho\cos\varphi \bullet \frac{1}{5}\rho^2 \bullet \rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi$$

$$= \left[-\frac{\cos^2\varphi}{2}\right]_0^{\pi/2}[2\pi]\left[\frac{\rho^6}{30}\right]_0^5 = \left[\frac{1}{2}\right][2\pi]\left[\frac{5^5}{6}\right] = \frac{5^5}{6}\pi = \frac{3125}{6}\pi$$

$$\bar{z} = \frac{z\text{-mom}}{M} = \frac{5^5\pi}{6 \bullet 250\pi} = \frac{25}{12}$$

7. A cardboard box is constructed with a hinge at the back so that the top, bottom and back have one sheet of cardboard while the sides and front have two sheets of cardboard. If the volume is 3 ft^3, find the dimensions of the box which minimize the amount of cardboard needed.

Minimize $A = 2xy + 3xz + 4yz$ subject to $V = xyz = 3$

METHOD 1: Eliminate a Variable

$z = \frac{3}{xy}$ $\quad A = 2xy + 3x\frac{3}{xy} + 4y\frac{3}{xy} = 2xy + \frac{9}{y} + \frac{12}{x}$

$A_x = 2y - \frac{12}{x^2} = 0$ $\quad A_y = 2x - \frac{9}{y^2} = 0$

$y = \frac{6}{x^2}$ $\quad 2x - \frac{9}{\left(\frac{6}{x^2}\right)^2} = 0$ $\quad 2x - \frac{9x^4}{36} = 0$ $\quad 2x - \frac{1}{4}x^4 = 0$

$\frac{1}{4}x^4 = 2x$ $\quad x^3 = 8$ $\quad x = 2$ $\quad y = \frac{6}{x^2} = \frac{6}{4} = \frac{3}{2}$ $\quad z = \frac{3}{xy} = \frac{3}{2\left(\frac{3}{2}\right)} = 1$

METHOD 2: Lagrange Multipliers

$\vec{\nabla}A = (2y + 3z, 2x + 4z, 3x + 4y)$ $\quad \vec{\nabla}V = (yz, xz, xy)$ $\quad \vec{\nabla}A = \lambda\vec{\nabla}V$

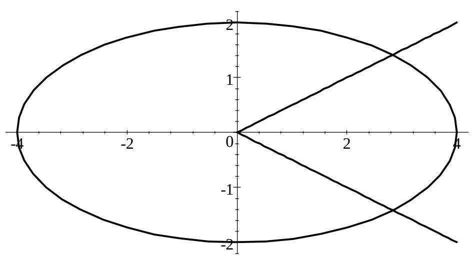
$2y + 3z = \lambda yz$ $\quad 2x + 4z = \lambda xz$ $\quad 3x + 4y = \lambda xy$

(1) $\lambda = \frac{2y + 3z}{yz} = \frac{2}{z} + \frac{3}{y}$ $\quad$ (2) $\lambda = \frac{2x + 4z}{xz} = \frac{2}{z} + \frac{4}{x}$ $\quad$ (3) $\lambda = \frac{3x + 4y}{xy} = \frac{3}{y} + \frac{4}{x}$

(1)=(2): $\frac{3}{y} = \frac{4}{x}$ $\quad x = \frac{4}{3}y$ $\quad$ (2)=(3): $\frac{2}{z} = \frac{3}{y}$ $\quad z = \frac{2}{3}y$

$V = 3 = xyz = \left(\frac{4}{3}y\right)y\left(\frac{2}{3}y\right) = \frac{8}{9}y^3$ $\quad y^3 = \frac{27}{8}$ $\quad y = \frac{3}{2}$ $\quad x = 2$ $\quad z = 1$

8. Compute $\iint_R x\,dx\,dy$ over the region inside the ellipse $\frac{x^2}{16} + \frac{y^2}{4} = 1$ between the lines $y = \frac{x}{2}$ and $y = -\frac{x}{2}$ in the 1st and 4th quadrants.



HINT:

Use the elliptic coordinate system:

$x = 4t\cos\theta \qquad y = 2t\sin\theta$

Find the Jacobian: $J = \left\| \begin{matrix} 4\cos\theta & 2\sin\theta \\ -4t\sin\theta & 2t\cos\theta \end{matrix} \right\| = \left|8t\cos^2\theta - -8t\sin^2\theta\right| = 8t$

Find the limits:

$\frac{x^2}{16} + \frac{y^2}{4} = 1$ $\quad \frac{16t^2\cos^2\theta}{16} + \frac{2t^2\sin^2\theta}{4} = 1$ $\quad t^2 = 1$ $\quad 0 \le t \le 1$

$y = \pm\frac{x}{2}$ $\quad 2t\sin\theta = \pm\frac{4t\cos\theta}{2}$ $\quad \tan\theta = \pm 1$ $\quad -\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$

Find the integrand: $\quad x = 4t\cos\theta$

$$\iint_R x\,dx\,dy = \int_{-\pi/4}^{\pi/4}\int_0^1 4t\cos\theta \bullet 8t\,dt\,d\theta = 32\Big[\sin\theta\Big]_{\theta=-\pi/4}^{\pi/4}\left[\frac{t^3}{3}\right]_{t=0}^{1} = 32\left[\frac{1}{\sqrt{2}} - \frac{-1}{\sqrt{2}}\right]\frac{1}{3}$$

$$= \frac{64}{3\sqrt{2}} = \frac{32\sqrt{2}}{3} = 15.085$$