MATH 253 Honors EXAM 3 Fall 1999

Sections 201-202 Solutions P. Yasskin

Multiple Choice: (8 points each)

1. Compute the line integral $\int_{\substack{A\\H}}^{B} \vec{F} \bullet d\vec{s}$ of the vector field $\vec{F} = (yz, -xz, z)$ along the helix H parametrized by $\vec{r}(t) = (3\cos t, 3\sin t, 4t)$ between $A = (3,0,0)$ and $B = (-3,0,4\pi)$.

a. 20π

b. 26π

c. $26\pi^2$

d. -20π

e. $-10\pi^2$ correctchoice

$\vec{r}(t) = (3\cos t, 3\sin t, 4t) \qquad \vec{v}(t) = (-3\sin t, 3\cos t, 4) \qquad \vec{F}(\vec{r}(t)) = (12t\sin t, -12t\cos t, 4t)$

$\vec{r}(a) = (3\cos a, 3\sin a, 4a) = A = (3,0,0) \qquad \Rightarrow \qquad a = 0$

$\vec{r}(b) = (3\cos b, 3\sin b, 4b) = B = (-3,0,4\pi) \qquad \Rightarrow \qquad b = \pi$

$$\int_{\substack{A\\H}}^{B} \vec{F} \bullet d\vec{s} = \int_0^\pi \vec{F} \bullet \vec{v}\,dt = \int_0^\pi \left(-36t\sin^2 t - 36t\cos^2 t + 16t\right) dt = \int_0^\pi (-36t + 16t)\,dt$$

$$= \int_0^\pi (-20t)\,dt = \left[-10t^2\right]_0^\pi = -10\pi^2$$

2. Find the total mass of the helix H parametrized by $\vec{r}(t) = (3\cos t, 3\sin t, 4t)$ between $A = (3,0,0)$ and $B = (-3,0,4\pi)$ if the linear mass density is $\rho = z^2$.

a. $\dfrac{16\pi^3}{3}$

b. $\dfrac{40\pi^3}{3}$

c. $\dfrac{80\pi^3}{3}$ correctchoice

d. $16\pi^3$

e. $80\pi^3$

$\vec{r}(t) = (3\cos t, 3\sin t, 4t) \qquad \vec{v}(t) = (-3\sin t, 3\cos t, 4) \qquad |\vec{v}| = \sqrt{9\sin^2 t + 9\cos^2 t + 16} = 5$

$\rho = z^2 = 16t^2$

$$M = \int_0^\pi \rho\,ds = \int_0^\pi \rho\,|\vec{v}|\,dt = \int_0^\pi 16t^2\,5\,dt = 80\frac{t^3}{3}\Big|_0^\pi = \frac{80\pi^3}{3}$$

3. Compute $\oint(-xy^2\,dx + x^2y\,dy)$ counterclockwise around the complete boundary of the triangle whose vertices are $(0,0)$, $(2,4)$ and $(0,4)$. HINT: Use Green's Theorem.

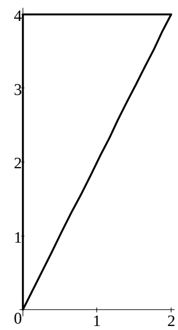
a. 4

b. 8

c. 16

d. 32 correctchoice

e. 64



$P = -xy^2 \qquad Q = x^2y \qquad \partial_x Q - \partial_y P = 2xy - -2xy = 4xy$

$$\oint(-xy^2\,dx + x^2y\,dy) = \oint(P\,dx + Q\,dy) = \iint \partial_x Q - \partial_y P\,dx\,dy$$

$$= \int_0^2 \int_{2x}^4 4xy\,dy\,dx = \int_0^2 \Big[2xy^2 \Big]_{y=2x}^4 dx = \int_0^2 (2x16 - 2x4x^2)\,dx$$

$$= \int_0^2 (32x - 8x^3)\,dx = \Big[16x^2 - 2x^4 \Big]_0^2 = 64 - 32 = 32$$

4. Compute $\int(yz\,dx + xz\,dy + xy\,dz)$ along the curve $\vec{r}(t) = \left(e^{\sin 4t}, \cos 5t, \ln\left(1 + \frac{t}{\pi}\right)\right)$ between $t = 0$ and $t = \pi$. HINT: Find a scalar potential for $\vec{F} = (yz, xz, xy)$.

a. $-\ln 2$ correctchoice

b. $1 - \ln 2$

c. $-1 - \ln 2$

d. $1 + \ln 2$

e. $-1 + \ln 2$

$A = \vec{r}(0) = (e^{\sin 0}, \cos 0, \ln(1)) = (1, 1, 0)$

$B = \vec{r}(\pi) = \left(e^{\sin 4\pi}, \cos 5\pi, \ln\left(1 + \frac{\pi}{\pi}\right)\right) = (1, -1, \ln 2)$

The scalar potential is $f = xyz$ since $\vec{\nabla} f = (yz, xz, xy) = \vec{F}$

By the Fundamental Theorem of Calculus for Curves:

$$\int(yz\,dx + xz\,dy + xy\,dz) = \int_A^B \vec{F} \bullet d\vec{s}$$

$$= \int_A^B \vec{\nabla} f \bullet d\vec{s} = f(B) - f(A) = (1)(-1)(\ln 2) - (1)(1)(0) = -\ln 2$$

5. (25 points) Stokes' Theorem states that if S is a surface in 3-space and ∂S is its boundary curve traversed counterclockwise as seen from the tip of the normal to S then

$$\iint_S \vec{\nabla} \times \vec{F} \bullet d\vec{S} = \oint_{\partial S} \vec{F} \bullet d\vec{s}$$

Verify Stokes' Theorem if

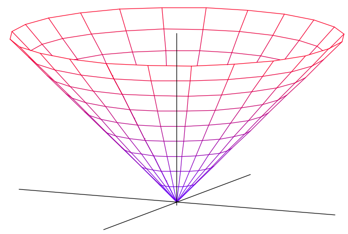
$F = (y, -x, x^2 + y^2)$

and S is the cone $z = \sqrt{x^2 + y^2}$ for $z \le 2$

with **normal pointing up and in**.

The cone may be parametrized by:

$\vec{R}(r, \theta) = (r\cos\theta, r\sin\theta, r)$



5a. (16 points) Compute $\iint_S \vec{\nabla} \times \vec{F} \bullet d\vec{S}$ using the following steps:

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ y & -x & x^2 + y^2 \end{vmatrix} = i(2y - 0) - j(2x - 0) + k(-1 - 1) = (2y, -2x, -2)$$

$\vec{R}_r = (\cos\theta, \sin\theta, 1)$

$\vec{R}_\theta = (-r\sin\theta, r\cos\theta, 0)$

$\vec{N} = \vec{R}_r \times \vec{R}_\theta = i(-r\cos\theta) - j(r\sin\theta) + k\left(r\cos^2\theta + r\sin^2\theta\right) = (-r\cos\theta, -r\sin\theta, r)$

This is oriented correctly as up and in.

$\left(\vec{\nabla} \times \vec{F}\right)\left(\vec{R}(r, \theta)\right) = (2r\sin\theta, -2r\cos\theta, -2)$

$$\iint_S \vec{\nabla} \times \vec{F} \bullet d\vec{S} = \iint \vec{\nabla} \times \vec{F} \bullet \vec{N}\, dr\, d\theta = \int\int (-2r^2 \sin\theta\cos\theta + 2r^2 \sin\theta\cos\theta - 2r)\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^2 (-2r)\, dr\, d\theta = 2\pi\left[-r^2\right]_0^2 = -8\pi$$

5b (9 points) Recall $F = (y, -x, x^2 + y^2)$ and S is the cone $z = \sqrt{x^2 + y^2}$ with **normal pointing up and in**.

Compute $\oint_{\partial S} \vec{F} \bullet d\vec{s}$ using the following steps:

(Remember to check the orientation of the curve.)

$\vec{r}(\theta) = (2\cos\theta, 2\sin\theta, 2)$

$\vec{v}(\theta) = (-2\sin\theta, 2\cos\theta, 0)$

$\vec{F}\left(\vec{r}(\theta)\right) = (2\sin\theta, -2\cos\theta, 4)$

$$\oint_{\partial S} \vec{F} \bullet d\vec{s} = \int_0^{2\pi} \vec{F} \bullet \vec{v}\, d\theta = \int_0^{2\pi} \left(-4\sin^2\theta - 4\cos^2\theta\right) d\theta = \int_0^{2\pi} (-4)\, d\theta = -8\pi$$

6. (25 points) Gauss' Theorem states that if V is a solid region and ∂V is its boundary surface with **outward normal** then

$$\iiint_V \vec{\nabla} \bullet \vec{F}\, dV = \iint_{\partial V} \vec{F} \bullet d\vec{S}$$

Verify Gauss' Theorem if

$F = (xz, yz, x^2 + y^2)$

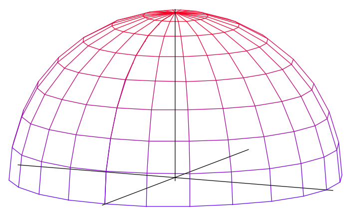
and V is the solid hemisphere

$0 \le z \le \sqrt{4 - x^2 - y^2}$.

Notice that ∂V consists of two parts:

the hemisphere H: $\quad z = \sqrt{4 - x^2 - y^2}$

and a disk D: $\quad x^2 + y^2 \le 4 \quad$ with $\quad z = 0$



6a. (5 pts) Compute $\displaystyle\iiint_V \vec{\nabla} \bullet \vec{F}\, dV$.

$\vec{\nabla} \bullet \vec{F} = 2z = 2\rho\cos\varphi$

$$\iiint_V \vec{\nabla} \bullet \vec{F}\, dV = \int_0^{\pi/2}\int_0^{2\pi}\int_0^2 2\rho\cos\varphi \bullet \rho^2 \sin\varphi\, d\rho\, d\theta\, d\varphi = (2\pi)\left[\frac{\sin^2\varphi}{2}\right]_0^{\pi/2}\left[\frac{2\rho^4}{4}\right]_0^2 = 8\pi$$

6b. (8 pts) Compute $\displaystyle\iint_D \vec{F} \bullet d\vec{S}$. (HINT: You parametrize the disk.)

$\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, 0)$

$\vec{R}_r = (\cos\theta, \sin\theta, 0)$

$\vec{R}_\theta = (-r\sin\theta, r\cos\theta, 0)$

$\vec{N} = (0,0,r)$ This is upward. Reverse it: $\vec{N} = (0,0,-r)$

$\vec{F}\left(\vec{R}(r,\theta)\right) = (xz, yz, x^2 + y^2) = (0,0,r^2)$

$$\iint_D \vec{F} \bullet d\vec{S} = \iint_D \vec{F} \bullet \vec{N}\, dr\, d\theta = \int_0^{2\pi}\int_0^2 (-r^3)\, dr\, d\theta = -2\pi\left[\frac{r^4}{4}\right]_0^2 = -8\pi$$

6c. (9 pts) Compute $\displaystyle\iint_H \vec{F} \bullet d\vec{S}$ over the hemisphere parametrized by

$\vec{R}(\varphi,\theta) = (2\sin\varphi\cos\theta, 2\sin\varphi\sin\theta, 2\cos\varphi)$

$\vec{R}_\varphi = (2\cos\varphi\cos\theta, 2\cos\varphi\sin\theta, -2\sin\varphi)$

$\vec{R}_\theta = (-2\sin\varphi\sin\theta, 2\sin\varphi\cos\theta, 0)$

$\vec{N} = i\left(4\sin^2\varphi\cos\theta\right) - j\left(-4\sin^2\varphi\sin\theta\right) + k\left(4\sin\varphi\cos\varphi\left[\cos^2\theta + \sin^2\theta\right]\right)$

$\quad = \left(4\sin^2\varphi\cos\theta, 4\sin^2\varphi\sin\theta, 4\sin\varphi\cos\varphi\right)$ This is outward.

$\vec{F}\left(\vec{R}(\varphi,\theta)\right) = (xz, yz, x^2 + y^2) = \left(4\sin\varphi\cos\varphi\cos\theta, 4\sin\varphi\cos\varphi\sin\theta, 4\sin^2\varphi\right)$

$\vec{F} \bullet \vec{N} = 16\sin^3\varphi\cos\varphi\cos^2\theta + 16\sin^3\varphi\cos\varphi\sin^2\theta + 16\sin^3\varphi\cos\varphi = 32\sin^3\varphi\cos\varphi$

$$\iint_H \vec{F} \bullet d\vec{S} = \int_0^{2\pi}\int_0^{\pi/2} 32\sin^3\varphi\cos\varphi\, d\varphi\, d\theta = 2\pi \bullet 32\left[\frac{\sin^4\varphi}{4}\right]_0^{\pi/2} = 16\pi$$

6d. (3 pts) Combine $\iint_H \vec{F} \bullet d\vec{S}$ and $\iint_D \vec{F} \bullet d\vec{S}$. to obtain $\iint_{\partial V} \vec{F} \bullet d\vec{S}$.

Be sure to discuss the orientations of the surfaces (here or above) and give a formula before you plug in numbers.

The normals must point outward. $\vec{N}_H$ points up. We reversed $\vec{N}_D$ so it points downward.

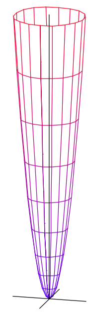
$\iint_{\partial V} \vec{F} \bullet d\vec{S} = \iint_H \vec{F} \bullet d\vec{S} + \iint_D \vec{F} \bullet d\vec{S} = 16\pi - 8\pi = 8\pi$ which agrees with (6a).

7. (20 points) The paraboloid at the right is the graph of the equation $z = 4x^2 + 4y^2$.

It may be parametrized as

$$\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, 4r^2).$$

Find the area of the paraboloid for $z \le 16$.



$$\vec{R}_r = (\cos\theta, \sin\theta, 8r)$$

$$\vec{R}_\theta = (-r\sin\theta, r\cos\theta, 0)$$

$$\vec{N} = i(-8r^2\cos\theta) - j(8r^2\sin\theta) + k\left(r\cos^2\theta + r\sin^2\theta\right) = (-8r^2\cos\theta, -8r^2\sin\theta, r)$$

$$\left|\vec{N}\right| = \sqrt{64r^4\cos^2\theta + 64r^4\sin^2\theta + r^2} = \sqrt{64r^4 + r^2} = r\sqrt{64r^2 + 1}$$

$$A = \iint \left|\vec{N}\right| \, dr\, d\theta = \int_0^{2\pi}\int_0^2 r\sqrt{64r^2+1}\, dr\, d\theta = 2\pi\left[\frac{2(64r^2+1)^{3/2}}{3 \bullet 128}\right]_0^2 = \frac{\pi}{96}(257^{3/2} - 1)$$