MATH 253 Honors FINAL EXAM Fall 1999

Sections 201-202 Solutions P. Yasskin

Multiple Choice: (7 points each)

1. Consider the line through the point $P=(4,4,4)$ which is perpendicular to the plane $x+2y+3z=7.$ Its tangent vector is

a. $(3,2,1)$
b. $(1,2,3)$ correctchoice
c. $(7,6,5)$
d. $(5,6,7)$
e. $(4,4,4)$

If the equation of a plane is $Ax+By+Cz=D$ then the normal is $\vec{N}=(A,B,C)$. In this case, $\vec{N}=(1,2,3)$. Since the line is perpendicular to the plane, then its tangent vector is the normal to the plane. So, $\vec{v}=(1,2,3)$.

2. Find the plane tangent to the hyperbolic paraboloid $x-yz=0$ at the point $P=(6,3,2)$. Which of the following points does **not** lie on this plane?

a. $(-6,0,0)$
b. $(0,3,0)$
c. $(0,0,2)$
d. $(1,-1,-1)$ correctchoice
e. $(-1,1,1)$

The hyperbolic paraboloid is a level surface of the function $g=x-yz$. Its gradient is $\vec{\nabla}g=(1,-z,-y)$. So the normal to the surface at P is $\vec{N}=\vec{\nabla}g\Big|_{(6,3,2)}=(1,-2,-3)$. So the tangent plane is $\vec{N}\bullet X=\vec{N}\bullet P$, or $x-2y-3z=6-2\bullet 3-3\bullet 2=-6$. Plugging in each point, we find $(1,-1,-1)$ is not a solution.

3. Duke Skywater is flying the Millenium Eagle through a polaron field. His galactic coordinates are $(2300,4200,1600)$ measured in lightseconds and his velocity is $\vec{v}=(.2,.3,.4)$ measured in lightseconds per second. He measures the strength of the polaron field is $p=274$ milliwookies and its gradient is $\vec{\nabla}p=(3,2,2)$ milliwookies per lightsecond. Assuming a linear approximation for the polaron field and that his velocity is constant, how many seconds will Duke need to wait until the polaron field has grown to 286 milliwookies?

a. 2
b. 3
c. 4
d. 6 correctchoice
e. 12

The derivative along Duke's path is

$$\frac{dp}{dt} = \vec{v} \bullet \vec{\nabla} p = (.2, .3, .4) \frac{\text{lightseconds}}{\text{second}} \bullet (3, 2, 2) \frac{\text{milliwookies}}{\text{lightsecond}}$$

$$= .6 + .6 + .8 = 2 \frac{\text{milliwookies}}{\text{second}}$$

So the polaron field increases 2 milliwookies each second. To increase 12 milliwookies, it will take 6 seconds.

4. Consider the surface S parametrized by $\vec{R}(u,v) = (u+v, u-v, uv)$ for $0 \le u \le 2$ and $0 \le v \le 4$. Compute $\iint_S \vec{F} \bullet d\vec{S}$ where $\vec{F} = (y, x, y)$.

- **a**. -32
- **b**. -16
- **c**. 16 correctchoice
- **d**. 32
- **e**. 64

$\vec{R}_u = (1, 1, v) \qquad \vec{R}_v = (1, -1, u) \qquad \vec{N} = \vec{R}_u \times \vec{R}_v = (u+v, v-u, -2)$

$\vec{F} = (y, x, y) = (u-v, u+v, u-v)$

$\vec{F} \bullet \vec{N} = (u-v)(u+v) + (u+v)(v-u) - 2(u-v) = -2u + 2v$

$$\iint_S \vec{F} \bullet d\vec{S} = \iint_S \vec{F} \bullet \vec{N}\, du\, dv = \int_0^4 \int_0^2 -2u + 2v\, du\, dv = \int_0^4 \left[-u^2 + 2vu \right]_{u=0}^2 dv$$

$$= \int_0^4 \left[-4 + 4v \right] dv = \left[-4v + 2v^2 \right]_{v=0}^4 = -16 + 32 = 16$$

5. Consider the surface S parametrized by $\vec{R}(u,v) = (u+v, u-v, uv)$. Find the plane tangent to this surface at the point $P = \vec{R}(1,2) = (3, -1, 2)$. Which of the following points does **not** lie on this plane?

- **a**. $(3, 0, 0)$ correctchoice
- **b**. $(0, 4, 0)$
- **c**. $(0, 0, -2)$
- **d**. $(1, 1, 0)$
- **e**. $(0, 6, 1)$

$\vec{R}_u = (1, 1, v) \qquad \vec{R}_v = (1, -1, u) \qquad \vec{N} = \vec{R}_u \times \vec{R}_v = (u+v, v-u, -2)$

The normal at P is $\vec{N}_P = \vec{N}_{(1,2)} = (3, 1, -2)$ and the tangent plane is $\vec{N} \bullet X = \vec{N} \bullet P$, or $3x + y - 2z = 3(3) + (-1) - 2(2) = 4$. Plugging in each point, we find $(3, 0, 0)$ is not a solution.

6. Compute $\oint(-x^2y^2\,dx + 2xy^3\,dy)$ over the complete boundary of the semicircular area $0 \le y \le \sqrt{4-x^2}$ traversed counterclockwise.

a. 0
b. 16
c. $\dfrac{4}{5}$
d. $\dfrac{80}{5}$
e. $\dfrac{128}{5}$ correctchoice

By Green's Theorem:

$$\oint(-x^2y^2\,dx + 2xy^3\,dy) = \iint \frac{\partial}{\partial x}(2xy^3) - \frac{\partial}{\partial y}(-x^2y^2)\,dx\,dy = \iint(2y^3 + 2x^2y)\,dx\,dy$$

$$= \iint 2(y^2 + x^2)y\,dx\,dy = \int_0^{\pi}\int_0^2 2r^2 r\sin\theta\, r\,dr\,d\theta = 2\Big[-\cos\theta\Big]_0^{\pi}\left[\frac{r^5}{5}\right]_0^2 = \frac{128}{5}$$

7. Compute $\iint\limits_S \frac{x^3z^2}{3}\,dy\,dz + \frac{y^3z^2}{3}\,dz\,dx + \frac{z^5}{5}\,dx\,dy$ over the complete surface of the sphere $x^2 + y^2 + z^2 = 4$ with outward normal.

a. $\dfrac{512\pi}{21}$ correctchoice
b. $\dfrac{32\pi^2}{5}$
c. $\dfrac{128\pi}{5}$
d. $\dfrac{16\pi}{3}$
e. $\dfrac{256\pi}{15}$

Apply Gauss' Theorem in spherical coordinates:

$$\vec{F} = \left(\frac{x^3z^2}{3}, \frac{y^3z^2}{3}, \frac{z^5}{5}\right) \qquad \vec{\nabla}\bullet\vec{F} = x^2z^2 + y^2z^2 + z^4 = (x^2+y^2+z^2)z^2 = \rho^2\bullet\rho^2\cos^2\theta$$

$$I = \iiint \vec{\nabla}\bullet\vec{F}\,dV = \int_0^{2\pi}\int_0^{\pi}\int_0^2 \rho^4\cos^2\theta\bullet\rho^2\sin\theta\,d\rho\,d\varphi\,d\theta = 2\pi\left[-\frac{\cos^3\theta}{3}\right]_0^{\pi}\left[\frac{\rho^7}{7}\right]_0^2 = \frac{512\pi}{21}$$

8. (15 points) Find the point in the first octant on the surface $z = \dfrac{32}{x^4y^2}$ which is closest to the origin.

Minimize $f = x^2 + y^2 + z^2$ on the surface $g = zx^4y^2 = 32$.

$$\vec{\nabla}f = (2x, 2y, 2z) \qquad \vec{\nabla}g = (4zx^3y^2, 2zx^4y, x^4y^2) \qquad \vec{\nabla}f = \lambda\vec{\nabla}g$$

$$2x = \lambda 4zx^3y^2 \qquad 2y = \lambda 2zx^4y \qquad 2z = \lambda x^4y^2 \qquad \lambda = \frac{1}{2zx^2y^2} = \frac{1}{zx^4} = \frac{2z}{x^4y^2}$$

$$x^2 = 2y^2 \qquad y^2 = 2z^2 \qquad x = \sqrt{2}\,y \qquad z = \frac{1}{\sqrt{2}}y$$

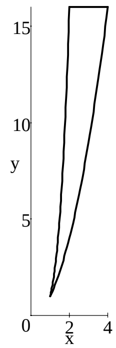
$$g = zx^4y^2 = \left(\frac{1}{\sqrt{2}}y\right)\left(\sqrt{2}\,y\right)^4 y^2 = 2^{3/2}y^7 = 32 = 2^5 \qquad y^7 = 2^{7/2}$$

$$y = \sqrt{2} \qquad x = 2 \qquad z = 1 \qquad (x,y,z) = \left(2, \sqrt{2}, 1\right)$$

9. (10 points) Compute $\iint_R x\,dA$

over the region R in the first quadrant bounded by the curves

$y = x^2$, $y = x^4$ and $y = 16$.

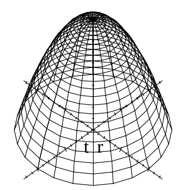


The left edge is $y = x^4$ or $x = y^{1/4}$. The right edge is $y = x^2$ or $x = y^{1/2}$.

$$\iint_R x\,dA = \int_1^{16}\int_{y^{1/4}}^{y^{1/2}} x\,dx\,dy = \int_1^{16}\left[\frac{x^2}{2}\right]_{x=y^{1/4}}^{y^{1/2}} dy = \int_1^{16}\left[\frac{y}{2} - \frac{y^{1/2}}{2}\right] dy$$

$$= \left[\frac{y^2}{4} - \frac{y^{3/2}}{3}\right]_{y=1}^{16} = \left[\frac{256}{4} - \frac{64}{3}\right] - \left[\frac{1}{4} - \frac{1}{3}\right] = \frac{255}{4} - 21 = \frac{171}{4}$$

10. (15 points) Find the mass and center of mass of the solid below the paraboloid $z = 4 - x^2 - y^2$ above the xy-plane, if the density is $\delta = x^2 + y^2$. (11 points for setting up the integrals and the final formula.)



In cylindrical coordinates, the paraboloid is $z = 4 - r^2$, the density is $\delta = r^2$ and the Jacobian is r.

$$M = \iiint \delta\,dV = \int_0^{2\pi}\int_0^2\int_0^{4-r^2} r^2\,r\,dz\,dr\,d\theta$$

$$= 2\pi\int_0^2 r^3 z\Big|_{z=0}^{4-r^2} dr = 2\pi\int_0^2 r^3(4-r^2)\,dr$$

$$= 2\pi\left[r^4 - \frac{r^6}{6}\right]_0^2 = 2\pi\left(16 - \frac{32}{3}\right) = \frac{32\pi}{3}$$

$$z\text{-mom} = \iiint z\delta\,dV = \int_0^{2\pi}\int_0^2\int_0^{4-r^2} z r^3\,dz\,dr\,d\theta$$

$$= 2\pi\int_0^2 r^3\frac{z^2}{2}\Big|_{z=0}^{4-r^2} dr = \pi\int_0^2 r^3(4-r^2)^2\,dr$$

Let $u = r^2$. Then $du = 2r\,dr$ and $r\,dr = \frac{1}{2}du$. So

$$z\text{-mom} = \frac{\pi}{2}\int_0^4 u(4-u)^2\,du = \frac{\pi}{2}\int_0^4 u(16 - 8u + u^2)\,du = \frac{\pi}{2}\left[8u^2 - 8\frac{u^3}{3} + \frac{u^4}{4}\right]_0^4$$

$$= \frac{\pi}{2}\left(128 - \frac{512}{3} + 64\right) = \frac{32\pi}{3}$$

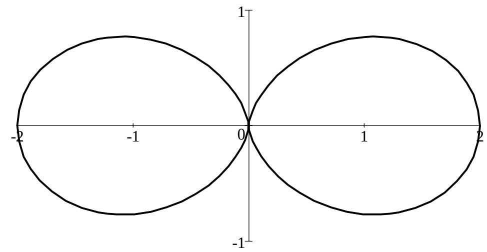
$$\bar{z} = \frac{z\text{-mom}}{M} = \frac{32\pi}{3}\,\frac{3}{32\pi} = 1$$

$\bar{x} = \bar{y} = 0$ by symmetry.

11. (15 points) Find the area and centroid of the **right** leaf of the rose

$$r = 2\cos^2\theta.$$

(12 points for setting up the integrals and the final formula.)



$$A = \iint 1\, dA = \int_{-\pi/2}^{\pi/2} \int_0^{2\cos^2\theta} r\, dr\, d\theta$$
$$= \int_{-\pi/2}^{\pi/2} \left[\frac{r^2}{2} \right]_{r=0}^{2\cos^2\theta} d\theta = \int_{-\pi/2}^{\pi/2} 2\cos^4\theta\, d\theta = \int_{-\pi/2}^{\pi/2} 2\left(\frac{1+\cos 2\theta}{2} \right)^2 d\theta$$
$$= \frac{1}{2} \int_{-\pi/2}^{\pi/2} (1 + 2\cos 2\theta + \cos^2 2\theta)\, d\theta = \frac{1}{2} \int_{-\pi/2}^{\pi/2} \left(1 + 2\cos 2\theta + \frac{1+\cos 4\theta}{2} \right) d\theta$$
$$= \frac{1}{2} \int_{-\pi/2}^{\pi/2} \left(\frac{3}{2} + 2\cos 2\theta + \frac{1}{2}\cos 4\theta \right) d\theta = \frac{1}{2} \left[\frac{3}{2}\theta + \sin 2\theta + \frac{1}{8}\sin 4\theta \right]_{-\pi/2}^{\pi/2}$$
$$= \frac{3}{4} \left(\frac{\pi}{2} - -\frac{\pi}{2} \right) = \frac{3\pi}{4}$$

$$x\text{-mom} = \iint x\, dA = \int_{-\pi/2}^{\pi/2} \int_0^{2\cos^2\theta} r^2 \cos\theta\, dr\, d\theta$$
$$= \int_{-\pi/2}^{\pi/2} \left[\frac{r^3}{3} \right]_{r=0}^{2\cos^2\theta} \cos\theta\, d\theta = \frac{8}{3} \int_{-\pi/2}^{\pi/2} \cos^6\theta \cos\theta\, d\theta$$
$$= \frac{8}{3} \int_{-\pi/2}^{\pi/2} \left(1 - \sin^2\theta\right)^3 \cos\theta\, d\theta = \frac{8}{3} \int_{-1}^{1} (1-u^2)^3\, du = \frac{8}{3} \int_{-1}^{1} (1 - 3u^2 + 3u^4 - u^6)\, du$$
$$= \frac{8}{3} \left[u - u^3 + \frac{3u^5}{5} - \frac{u^7}{7} \right]_{-1}^{1} = \frac{16}{3} \left[1 - 1 + \frac{3}{5} - \frac{1}{7} \right] = \frac{16}{3} \frac{16}{35} = \frac{256}{105}$$

$$\bar{x} = \frac{x\text{-mom}}{A} = \frac{256}{105} \frac{4}{3\pi} = \frac{1024}{315\pi}$$

$\bar{y} = 0$ by symmetry.