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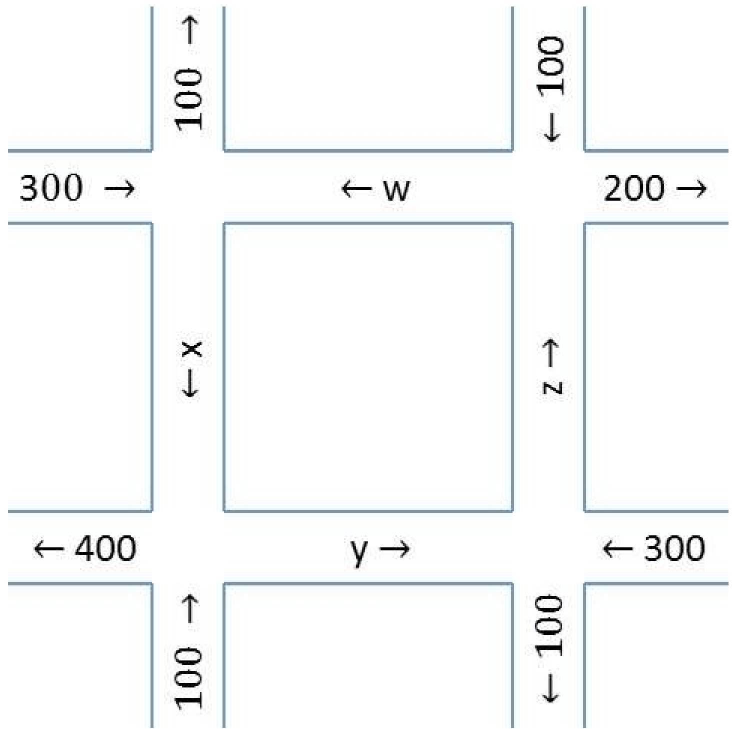
Math 304 Exam 1 Version A Spring 2017

Section 501 Solutions P. Yasskin

Points indicated. Show all work.

1	/20	3	/30
2	/45	4	/10
		Total	/105

1. (20 points) Consider the traffic flow system shown at the right.



a. Write out the equations for the system.
Write out the augmented matrix.
Keep the variables in the order $w, \; x, \; y, \; z$.
DO NOT SOLVE THE SYSTEM.

Solution: The equations are:

$$\begin{aligned} w+300 &= x+100 & \qquad w-x &= -200 \\ x+100 &= y+400 & \qquad x-y &= 300 \\ y+300 &= z+100 & \qquad y-z &= -200 \\ z+100 &= w+200 & \qquad -w+z &= 100 \end{aligned}$$

The augmented matrix is

$$\left(\begin{array}{cccc|c} 1 & -1 & 0 & 0 & -200 \\ 0 & 1 & -1 & 0 & 300 \\ 0 & 0 & 1 & -1 & -200 \\ -1 & 0 & 0 & 1 & 100 \end{array}\right)$$

b. Compute the determinant of the matrix of coefficients.

Expand on the first column.
Then use: "The determinant of a triangular matrix is the product of the diagonal entries."

$$\begin{vmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \end{vmatrix} = 1\begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{vmatrix} - -1\begin{vmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{vmatrix} = 1(1) + 1(-1) = 0$$

c. One solution is $w = 200$, $x = 400$, $y = 100$, $z = 300$. How many solutions are there?

Circle one:

Exactly 1 solution. Exactly 2 solutions. Exactly 4 solutions. Infinitely many solutions.

Since the determinant is zero, there are either no solutions or infinitely many solutions.

2. (45 points) Let $A = \begin{pmatrix} 1 & 2 & 0 & 1 & 0 \\ -1 & -2 & 3 & 8 & 3 \\ 2 & 4 & 0 & 2 & 1 \\ -1 & -2 & 1 & 2 & 1 \end{pmatrix}$.

a. Transform A into reduced row eschelon form. Call the result $rref(A)$. (Be sure to give reasons for each step.)

$$\begin{pmatrix} 1 & 2 & 0 & 1 & 0 \\ -1 & -2 & 3 & 8 & 3 \\ 2 & 4 & 0 & 2 & 1 \\ -1 & -2 & 1 & 2 & 1 \end{pmatrix} \begin{matrix} \\ R_2 + R_1 \\ R_3 - 2R_1 \\ R_4 + R_1 \end{matrix} \Rightarrow \begin{pmatrix} 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 3 & 9 & 3 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 3 & 1 \end{pmatrix} \begin{matrix} \\ \frac{1}{3}R_2 \\ \\ \\ \end{matrix} \Rightarrow \begin{pmatrix} 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 3 & 1 \end{pmatrix} \begin{matrix} \\ \\ \\ R_4 - R_2 \end{matrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{matrix} \\ R_2 - R_3 \\ \\ \\ \end{matrix} \Rightarrow \begin{pmatrix} 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

b. How many leading 1's are there in $rref(A)$? #1's = 3

c. What are the dimensions of the **null space**, **column space** and **row space** of A?

$\dim(N(A)) =$ 2 $\quad \dim(Col(A)) =$ 3 $\quad \dim(Row(A)) =$ 3

$\dim(Col(A))$ and $\dim(Row(A))$ are the rank which is the number of leading 1's.

$\dim(N(A))$ is the nullity which is the number of free variables in the solution of $A\vec{x} = \vec{0}$, which is the number of columns without leading 1's.

d. Find a basis for $Col(A)$.

Short answer: A basis is the columns in the original matrix A which match the columns with leading 1's in $rref(A)$. So a basis is

$$A_1 = \begin{pmatrix} 1 \\ -1 \\ 2 \\ -1 \end{pmatrix} \quad A_3 = \begin{pmatrix} 0 \\ 3 \\ 0 \\ 1 \end{pmatrix} \quad A_5 = \begin{pmatrix} 0 \\ 3 \\ 1 \\ 1 \end{pmatrix}.$$

Long answer: $Col(A) = Span(A_1, A_2, A_3, A_4, A_5)$. To check linear independence,

we solve $x_1A_1 + x_2A_2 + x_3A_3 + x_4A_4 + x_5A_5 = \vec{0}$. From $rref(A)$, the solution is

$$\begin{aligned} x_1 &= -2r - s \\ x_2 &= r \\ x_3 &= -3s \\ x_4 &= s \\ x_5 &= 0 \end{aligned}.$$

If we set $\begin{matrix} r = 1 \\ s = 0 \end{matrix}$, the solution is $\begin{matrix} x_1 = -2 \\ x_2 = 1 \\ x_3 = 0 \\ x_4 = 0 \\ x_5 = 0 \end{matrix}$ which says $-2A_1 + A_2 = \vec{0}$ or $A_2 = 2A_1$.

If we set $\begin{matrix} r = 0 \\ s = 1 \end{matrix}$, the solution is $\begin{matrix} x_1 = -1 \\ x_2 = 0 \\ x_3 = -3 \\ x_4 = 1 \\ x_5 = 0 \end{matrix}$ which says $-A_1 - 3A_3 + A_4 = \vec{0}$ or $A_4 = A_1 + 3A_3$.

So $Col(A) = Span(A_1, A_3, A_5)$ and the basis is A_1, A_3, A_5

e. Find a basis for $Row(A)$.

Short answer: A basis is the rows in the matrix $rref(A)$ which have leading 1's. So a basis is $rref(A)^1 = \begin{pmatrix} 1 & 2 & 0 & 1 & 0 \end{pmatrix}$ $rref(A)^2 = \begin{pmatrix} 0 & 0 & 1 & 3 & 0 \end{pmatrix}$ $rref(A)^3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \end{pmatrix}$.

Long answer:

$$\begin{aligned} Row(A) &= Span(A^1, A^2, A^3, A^4) = Span\left(rref(A)^1, rref(A)^2, rref(A)^3, rref(A)^4\right) \\ &= Span\left(rref(A)^1, rref(A)^2, rref(A)^3\right) \quad \text{since } rref(A)^4 = \vec{0}. \end{aligned}$$

So the basis is $rref(A)^1$, $rref(A)^2$, $rref(A)^3$

f. Find a basis for $N(A)$.

We solve $A\vec{x} = \vec{0}$. From $rref(A)$, the solution is $\begin{matrix} x_1 = -2r - s \\ x_2 = r \\ x_3 = -3s \\ x_4 = s \\ x_5 = 0 \end{matrix}$.

$$\text{So } N(A) = \left\{ \vec{x} = \begin{pmatrix} -2r - s \\ r \\ -3s \\ s \\ 0 \end{pmatrix} = r\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} -1 \\ 0 \\ -3 \\ 1 \\ 0 \end{pmatrix} \right\} = Span\left(\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ -3 \\ 1 \\ 0 \end{pmatrix} \right)$$

So a basis is $\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ -3 \\ 1 \\ 0 \end{pmatrix}$.

3. (30 points) Consider the vector space $P_3 = \{\text{polynomials of degree } < 3\}$. The standard basis is

$$e_1 = 1 \qquad e_2 = x \qquad e_3 = x^2$$

Let the f basis be

$$f_1 = 1 + x^2 \qquad f_2 = x + x^2 \qquad f_3 = x^2$$

Let the g basis be

$$g_1 = 1 \qquad g_2 = 1 + x \qquad g_3 = 1 + x^2$$

a. Find the change of basis matrix from the f basis to the e basis. Call it $\underset{e \leftarrow f}{C}$.

$$\begin{aligned} f_1 &= 1 + x^2 &&= 1e_1 + 0e_2 + 1e_3 \\ f_2 &= x + x^2 &&= 0e_1 + 1e_2 + 1e_3 \\ f_3 &= x^2 &&= 0e_1 + 0e_2 + 1e_3 \end{aligned} \qquad \underset{e \leftarrow f}{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

b. Find the change of basis matrix from the g basis to the e basis. Call it $\underset{e \leftarrow g}{C}$.

$$\begin{aligned} g_1 &= 1 &&= 1e_1 + 0e_2 + 0e_3 \\ g_2 &= 1 + x &&= 1e_1 + 1e_2 + 0e_3 \\ g_3 &= 1 + x^2 &&= 1e_1 + 0e_2 + 1e_3 \end{aligned} \qquad \underset{e \leftarrow g}{C} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

c. Find the change of basis matrix from the f basis to the g basis. Call it $\underset{g \leftarrow f}{C}$.

$$\underset{g \leftarrow f}{C} = \underset{g \leftarrow e}{C}\ \underset{e \leftarrow f}{C} = \left(\underset{e \leftarrow g}{C}\right)^{-1} \underset{e \leftarrow f}{C}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_1 - R_2 - R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \qquad \left(\underset{e \leftarrow g}{C}\right)^{-1} = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\underset{g \leftarrow f}{C} = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -2 & -1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

d. Use $\underset{g \leftarrow f}{C}$ to rewrite the polynomial $p = 6f_1 + 3f_2 - 2f_3$ in the g basis, i.e. find a, b, and c so that $p = ag_1 + bg_2 + cg_3$.

$$(p)_f = \begin{pmatrix} 6 \\ 3 \\ -2 \end{pmatrix} \qquad (p)_g = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \underset{g \leftarrow f}{C} (p)_f = \begin{pmatrix} 0 & -2 & -1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 6 \\ 3 \\ -2 \end{pmatrix} = \begin{pmatrix} -4 \\ 3 \\ 7 \end{pmatrix}$$

So $p = -4g_1 + 3g_2 + 7g_3$.

We check: $p = 6f_1 + 3f_2 - 2f_3 = 6(1 + x^2) + 3(x + x^2) - 2(x^2) = 7x^2 + 3x + 6$

$p = ag_1 + bg_2 + cg_3 = -4(1) + 3(1 + x) + 7(1 + x^2) = 7x^2 + 3x + 6$

4. $\left(10 \text{ points}\right)$ By definition, a matrix, A, is idempotent if $A^2 = A$.

a. Show if A is idempotent then $\mathbf{1} - A$ is also idempotent.

To show $\mathbf{1} - A$ is idempotent, we compute

$$(\mathbf{1} - A)^2 = \mathbf{1}^2 - \mathbf{1}A - A\mathbf{1} + A^2 = \mathbf{1} - 2A + A = (\mathbf{1} - A)$$

b. Show if A is idempotent then $\mathbf{1} + A$ is non-singular and $(\mathbf{1} + A)^{-1} = \mathbf{1} - \frac{1}{2}A$.

To show $(\mathbf{1} + A)^{-1} = \mathbf{1} - \frac{1}{2}A$, we compute

$$(\mathbf{1} + A)\left(\mathbf{1} - \frac{1}{2}A\right) = \mathbf{1}^2 + A\mathbf{1} - \frac{1}{2}\mathbf{1}A - \frac{1}{2}A^2 = 1 + A - \frac{1}{2}A - \frac{1}{2}A = 1$$

So $(\mathbf{1} + A)^{-1} = \mathbf{1} - \frac{1}{2}A$, and $\mathbf{1} + A$ is invertible, i.e. non-singular.