Name________________________________

Math 304 Exam 2 Spring 2017

Section 501 P. Yasskin

Points indicated. Show all work.

1	/20	3	/45
2	/20	4	/30
		Total	/115

1. $(20$ points$)$ Determine which of the following functions is linear and which is not.
If it is linear, prove it.
If it is not linear, show why it violates the definition.

a. $L : P_3 \to P_3 : L(p) = x + p'(x)$

b. $L : P_3 \to P_3 : L(p) = xp'(x)$

2. $\left(20\right.$ points$\left.\right)$ Let $M(2,2)$ be the vector space of 2×2 matrices.

Recall $(XY)_{ij} = \sum_{k=1}^{2} X_{ik} Y_{kj}$ and $tr(X) = \sum_{i=1}^{2} X_{ii} = X_{11} + X_{22}$

Determine which of the following is an inner product on $M(2,2)$ and which is not.
If it is an inner product, prove it.
If it is not an inner product, show why it violates the definition.

a. $\langle X, Y \rangle = tr(X^{\top} Y)$

b. $\langle X, Y \rangle = tr(X)\, tr(Y)$

3. (45 points) Let $V = Span(\sin^2\theta, \cos^2\theta)$ be the vector space of functions spanned by the basis functions $e_1 = \sin^2\theta$ and $e_2 = \cos^2\theta$. Here are some properties of these functions:

$$\frac{de_1}{d\theta} = 2\sin\theta\cos\theta \qquad \frac{de_2}{d\theta} = -2\cos\theta\sin\theta$$

$$\frac{d^2e_1}{d\theta^2} = 2\cos^2\theta - 2\sin^2\theta \qquad \frac{d^2e_2}{d\theta^2} = 2\sin^2\theta - 2\cos^2\theta$$

$$1 = \sin^2\theta + \cos^2\theta \qquad \cos 2\theta = \cos^2\theta - \sin^2\theta$$

a. Consider the linear operator $L : V \to V$ which computes second derivatives: $L(f) = \dfrac{d^2f}{d\theta^2}$.

Find the matrix of L relative to the (e_1, e_2) basis. Call it $\underset{e\leftarrow e}{A}$.

b. Another basis is $E_1 = 1$ and $E_2 = \cos 2\theta$.

Find the change of basis matrix from the E basis to the e basis. Call it $\underset{e\leftarrow E}{C}$.

Find the change of basis matrix from the e basis to the E basis. Call it $\underset{E\leftarrow e}{C}$..

c. Use the results of (a.) and (b.) to find the matrix of L relative to the (E_1, E_2) basis. Call it $\underset{E\leftarrow E}{B}$.

d. Using the matrix $\underset{E\leftarrow E}{B}$, what are $L(1) = L(E_1)$ and $L(\cos 2\theta) = L(E_2)$?

4. $(30$ points$)$ Let $V = Span(x, x^2)$ be the vector space spanned by the functions

$$v_1 = x \quad \text{and} \quad v_2 = x^2.$$

Use the inner product on V given by

$$\langle f, g \rangle = \int_0^1 fg\,dx$$

a. Find the angle between v_1 and v_2..

b. Start with the basis v_1 and v_2 and use the Gram-Schmidt procedure to produce an orthogonal basis w_1 and w_2 and an orthonormal basis u_1 and u_2.