Name________________________________

1	/20	3	/45
2	/20	4	/30
		Total	/115

Math 304 Exam 2 Spring 2017
Section 501 Solutions P. Yasskin

Points indicated. Show all work.

1. (20 points) Determine which of the following functions is linear and which is not.
If it is linear, prove it.
If it is not linear, show why it violates the definition.

a. $L : P_3 \to P_3 : L(p) = x + p'(x)$

Solution: Not Linear. First reason.
$L(p+q) = x + (p+q)'(x) = x + p'(x) + q'(x)$
$L(p) + L(q) = \left[x + p'\left(x\right)\right] + [x + q'(x)] = 2x + p'(x) + q'(x)$ These are not equal.
Alternate second reason:
$L(ap) = x + (ap)'(x) = x + ap'(x)$
$aL(p) = a[x + p'(x)] = ax + ap'(x)$ These are not equal.

b. $L : P_3 \to P_3 : L(p) = xp'(x)$

Solution: Yes Linear. Proof:
$L(p+q) = x(p+q)'(x) = x[p'(x) + q'(x)] = xp'(x) + xq'(x) = L(p) + L(q)$
$L(ap) = x(ap)'(x) = xa(p)'(x) = axp'(x) = aL(p)$

2. (20 points) Let $M(2,2)$ be the vector space of 2×2 matrices.
Recall $(XY)_{ij} = \sum_{k=1}^{2} X_{ik}Y_{kj}$ and $tr(X) = \sum_{i=1}^{2} X_{ii} = X_{11} + X_{22}$
Determine which of the following is an inner product on $M(2,2)$ and which is not.
If it is an inner product, prove it.
If it is not an inner product, show why it violates the definition.

a. $\langle X,Y\rangle = tr(X^\intercal Y)$

Solution: Yes Inner Product. Proof:
Since $tr(A) = tr(A^\intercal)$ and $(AB)^\intercal = B^\intercal A^\intercal$ and $tr(aA + bB) = atr(A) + btr(B)$ we have
$\langle Y,X\rangle = tr(Y^\intercal X) = tr((Y^\intercal X)^\intercal) = tr(X^\intercal Y^{\intercal\intercal}) = tr(X^\intercal Y) = \langle X,Y\rangle$ symmetric
$\langle aX + bY, Z\rangle = tr((aX + bY)^\intercal Z) = tr(aX^\intercal Z + bY^\intercal Z) = a\,tr(X^\intercal Z) + b\,tr(Y^\intercal Z) = a\langle X,Z\rangle + b\langle Y,Z\rangle$ linear
$\langle X,X\rangle = tr(X^\intercal X) = \sum_{i=1}^{2}(X^\intercal X)_{ii} = \sum_{i=1}^{2}\sum_{k=1}^{2} X^\intercal_{ik}X_{ki} = \sum_{i=1}^{2}\sum_{k=1}^{2} X_{ki}X_{ki} = X_{11}^2 + X_{12}^2 + X_{21}^2 + X_{22}^2 \geq 0$
and $= 0$ iff $X_{11} = X_{12} = X_{21} = X_{22} = 0$, i.e. $X = \mathbf{0}$. positive definite

b. $\langle X,Y\rangle = tr(X)\,tr(Y)$

Solution: Not Linear. Not positive definite:
Let $X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then $tr(X) = 1 - 1 = 0$ and so $\langle X,X\rangle = tr(X)\,tr(X) = 0$ but $X \neq \mathbf{0}$
Note: It is symmetric, linear and positive.

3. (45 points) Let $V = Span(\sin^2\theta, \cos^2\theta)$ be the vector space of functions spanned by the basis functions $e_1 = \sin^2\theta$ and $e_2 = \cos^2\theta$. Here are some properties of these functions:

$$\frac{de_1}{d\theta} = 2\sin\theta\cos\theta \qquad \frac{de_2}{d\theta} = -2\cos\theta\sin\theta$$
$$\frac{d^2e_1}{d\theta^2} = 2\cos^2\theta - 2\sin^2\theta \qquad \frac{d^2e_2}{d\theta^2} = 2\sin^2\theta - 2\cos^2\theta$$
$$1 = \sin^2\theta + \cos^2\theta \qquad \cos 2\theta = \cos^2\theta - \sin^2\theta$$

a. Consider the linear operator $L : V \to V$ which computes second derivatives: $L(f) = \frac{d^2f}{d\theta^2}$.

Find the matrix of L relative to the (e_1, e_2) basis. Call it $\underset{e\leftarrow e}{A}$.

Solution:

$$L(e_1) = \frac{d^2e_1}{d\theta^2} = 2\cos^2\theta - 2\sin^2\theta = -2e_1 + 2e_2$$
$$L(e_2) = \frac{d^2e_2}{d\theta^2} = 2\sin^2\theta - 2\cos^2\theta = 2e_1 - 2e_2$$
$$\underset{e\leftarrow e}{A} = \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}$$

b. Another basis is $E_1 = 1$ and $E_2 = \cos 2\theta$.

Find the change of basis matrix from the E basis to the e basis. Call it $\underset{e\leftarrow E}{C}$.

Find the change of basis matrix from the e basis to the E basis. Call it $\underset{E\leftarrow e}{C}$.

Solution:

$$E_1 = \sin^2\theta + \cos^2\theta = e_1 + e_2$$
$$E_2 = \cos^2\theta - \sin^2\theta = -e_1 + e_2$$
$$\underset{e\leftarrow E}{C} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$\underset{E\leftarrow e}{C} = \left(\underset{e\leftarrow E}{C}\right)^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \quad \text{since} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

c. Use the results of (a.) and (b.) to find the matrix of L relative to the (E_1, E_2) basis. Call it $\underset{E\leftarrow E}{B}$.

Solution:

$$\underset{E\leftarrow E}{B} = \underset{E\leftarrow e}{C}\ \underset{e\leftarrow e}{A}\ \underset{e\leftarrow E}{C} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -4 \end{pmatrix}$$

d. Using the matrix $\underset{E\leftarrow E}{B}$, what are $L(1) = L(E_1)$ and $L(\cos 2\theta) = L(E_2)$?

Solution: The entries in B are the coefficients of E_1 and E_2 in the formulas:

$$L(1) = L(E_1) = 0E_1 + 0E2 = 0$$
$$L(\cos 2\theta) = L(E_2) = 0E_1 - 4E_2 = -4\cos 2\theta$$

4. (30 points) Let $V = Span(x, x^2)$ be the vector space spanned by the functions

$$v_1 = x \quad \text{and} \quad v_2 = x^2.$$

Use the inner product on V given by

$$\langle f, g \rangle = \int_0^1 fg\,dx$$

a. Find the angle between v_1 and v_2.

Solution: $\langle v_1, v_2 \rangle = \int_0^1 v_1 v_2 \, d\theta = \int_0^1 x x^2 \, dx = \left[\frac{x^4}{4} \right]_0^1 = \frac{1}{4}$

$$\langle v_1, v_1 \rangle = \int_0^1 (v_1)^2 \, dx = \int_0^1 x^2 \, dx = \left[\frac{x^3}{3} \right]_0^1 = \frac{1}{3} \qquad |v_1| = \frac{1}{\sqrt{3}}$$

$$\langle v_2, v_2 \rangle = \int_0^1 (v_2)^2 \, dx = \int_0^1 (x^2)^2 \, dx = \left[\frac{x^5}{5} \right]_0^1 = \frac{1}{5} \qquad |v_2| = \frac{1}{\sqrt{5}}$$

$$\cos\theta = \frac{\langle v_1, v_2 \rangle}{|v_1||v_2|} = \frac{\frac{1}{4}}{\frac{1}{\sqrt{3}} \frac{1}{\sqrt{5}}} = \frac{\sqrt{3}\sqrt{5}}{4} = \frac{\sqrt{15}}{4} \qquad \theta = \arccos \frac{\sqrt{15}}{4}$$

b. Start with the basis v_1 and v_2 and use the Gram-Schmidt procedure to produce an orthogonal basis w_1 and w_2 and an orthonormal basis u_1 and u_2.

Solution: $w_1 = v_1 = x$

$$\langle w_1, w_1 \rangle = \int_0^1 (w_1)^2 \, dx = \int_0^1 x^2 \, dx = \left[\frac{x^3}{3} \right]_0^1 = \frac{1}{3} \qquad |w_1| = \frac{1}{\sqrt{3}}$$

$$u_1 = \frac{w_1}{|w_1|} = \sqrt{3}\,x$$

$$\langle v_2, w_1 \rangle = \int_0^1 v_2 w_1 \, d\theta = \int_0^1 x^2 x \, dx = \left[\frac{x^4}{4} \right]_0^1 = \frac{1}{4}$$

$$w_2 = v_2 - \frac{\langle v_2, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 = x^2 - \frac{\frac{1}{4}}{\frac{1}{3}} x = x^2 - \frac{3}{4} x$$

$$\langle w_2, w_2 \rangle = \int_0^1 (w_2)^2 \, dx = \int_0^1 \left(x^2 - \frac{3}{4} x \right)^2 dx = \int_0^1 \left(x^4 - \frac{3}{2} x^3 + \frac{9}{16} x^2 \right) dx$$

$$= \left[\frac{x^5}{5} - \frac{3}{2} \frac{x^4}{4} + \frac{9}{16} \frac{x^3}{3} \right]_0^1 = \frac{1}{5} - \frac{3}{8} + \frac{3}{16} = \frac{1}{80} \qquad |w_2| = \frac{1}{\sqrt{80}} = \frac{1}{4\sqrt{5}}$$

$$u_2 = \frac{w_2}{|w_2|} = 4\sqrt{5} \left(x^2 - \frac{3}{4} x \right)$$