Name________________________________

Math 304 Final Exam Spring 2017

Section 501 P. Yasskin

Points indicated. Show all work.

1	/20	3	/15
2	/32	4	/35
		Total	/102

You do not need to prove any basis is linearly independent in any problem.

1. $(20$ points$)$ Consider the vector space $P_3 = \{$polynomials of degree $< 3\}$.

a. Take the standard basis to be $e_1 = 1 \quad e_2 = x \quad e_3 = x^2$.
Find the components of $p = 2 + 3x + 4x^2$ relative to the e basis.

$p_e = \begin{pmatrix} \\ \\ \end{pmatrix}$

b. Another basis is $f_1 = 1 + x \quad f_2 = 1 + x^2 \quad f_3 = 2 + x$.
Find the change of basis matrix from the f basis to the e basis.

$\underset{e \leftarrow f}{C} = \begin{pmatrix} \\ \\ \end{pmatrix}$

c. Find the change of basis matrix from the e basis to the f basis.

$\underset{f \leftarrow e}{C} = \begin{pmatrix} \\ \\ \end{pmatrix}$

d. Find the components of $p = 2 + 3x + 4x^2$ relative to the f basis.

$p_f = \begin{pmatrix} \\ \\ \end{pmatrix}$

e. Find the polynomial q whose components relative to the f basis are $q_f = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$

Simplify fully.

$q =$

2. $\left(32 \text{ points}\right)$ Let $P_2 = \{\text{polynomials of degree } < 2\}$ and $P_3 = \{\text{polynomials of degree } < 3\}$.

Consider the linear map $L : P_2 \to P_3$ given by $L(p) = 2\int_1^x p\,dx$.

For example: $L(3+4x) = 2\int_1^x (3+4x)\,dx = 2[3x+2x^2]_1^x = 2(3x+2x^2-5) = -10+6x+4x^2$.

NOTE: $\{0\} = Span(0)$

a. Find the image of L. What is its dimension?
HINT: Take the general element of P_2 to be $p = a + bx$.

$\text{Im}(L) = Span\Big(\qquad\qquad\qquad\Big)$
$\dim \text{Im}(L) =$

b. Find the kernel of L. What is its dimension?

$Ker(L) = Span\Big(\qquad\qquad\qquad\Big)$
$\dim Ker(L) =$

c. Is L onto? Why?
Because:

Circle one:	
Yes	No

d. Is L one-to-one? Why?
Because:

Circle one:	
Yes	No

e. Find the matrix of L relative to the standard bases.

$e_1 = 1 \qquad e_2 = x \qquad\qquad\qquad$ for P_2

$E_1 = 1 \qquad E_2 = x \qquad E_3 = x^2 \qquad$ for P_3

$\underset{E \leftarrow e}{A} = \Big(\qquad\qquad\Big)$

(continued)

f. Find the null space of A. What is its dimension?

$Null(L) = Span\Big(\qquad \Big)$
$\dim Null(L) =$

g. Find the column space of A. What is its dimension?

$Col(L) = Span\Big(\qquad \Big)$
$\dim Col(L) =$

h. Find the row space of A. What is its dimension?

$Row(L) = Span\Big(\qquad \Big)$
$\dim Row(L) =$

3. (15 points) Consider the polynomial vector space $V = Span(x, x^2)$ with the inner product

$$\langle f, g \rangle = \int_0^1 \frac{fg}{x}\, dx$$

a. Find the angle between $v_1 = x$ and $v_2 = x^2$.

$\theta =$

b. Start with the basis $v_1 = x$ and $v_2 = x^2$ and use the Gram-Schmidt procedure to produce an orthogonal basis w_1 and w_2 and an orthonormal basis u_1 and u_2.

$w_1 =$

$u_1 =$

$w_2 =$

$u_2 =$

4. (35 points) Consider the matrix $A = \begin{pmatrix} 4 & 2 \\ -1 & 1 \end{pmatrix}$.

a. Find the eigenvalues of A. List them in ascending order.

$\lambda_1 =$ ______	$\lambda_2 =$ ______

b. Find the eigenvectors of A.

$\lambda_1 =$ ______:

$e_1 = \begin{pmatrix} \\ \end{pmatrix}$

$\lambda_2 =$ ______:

$e_2 = \begin{pmatrix} \\ \end{pmatrix}$

(continued)

Recall: $A = \begin{pmatrix} 4 & 2 \\ -1 & 1 \end{pmatrix}$.

c. Find a diagonal matrix D and an invertible matrix X so that $A = XDX^{-1}$.

$\underset{e\leftarrow e}{D} = \begin{pmatrix} & \\ & \end{pmatrix}$

$X = \begin{pmatrix} & \\ & \end{pmatrix}$

d. Find X^{-1}.

$X^{-1} = \begin{pmatrix} & \\ & \end{pmatrix}$

e. Compute $\cos(\pi A)$.

HINT: If $D = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$, then $\pi D = \begin{pmatrix} \alpha\pi & 0 \\ 0 & \beta\pi \end{pmatrix}$. What is $\cos(\pi D)$?

$\cos(\pi A) = \begin{pmatrix} & \\ & \end{pmatrix}$