Name

Math 304 Final Exam Spring 2017
Section 501 Solutions P. Yasskin

Points indicated. Show all work.

1	/20	3	/15
2	/32	4	/35
		Total	/102

You do not need to prove any basis is linearly independent in any problem.

1. $(20$ points$)$ Consider the vector space $P_3 = \{$polynomials of degree $< 3\}$.

a. Take the standard basis to be $e_1 = 1 \quad e_2 = x \quad e_3 = x^2$.
Find the components of $p = 2 + 3x + 4x^2$ relative to the e basis.

$$p = 2e_1 + 3e_2 + 4e_3$$

$$\boxed{p_e = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}}$$

b. Another basis is $f_1 = 1 + x \quad f_2 = 1 + x^2 \quad f_3 = 2 + x$.
Find the change of basis matrix from the f basis to the e basis.

$$\begin{aligned} f_1 &= 1 + x &&= 1e_1 + 1e_2 + 0e_3 \\ f_2 &= 1 + x^2 &&= 1e_1 + 0e_2 + 1e_3 \\ f_3 &= 2 + x &&= 2e_1 + 1e_2 + 0e_3 \end{aligned}$$

$$\boxed{\underset{e\leftarrow f}{C} = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}}$$

c. Find the change of basis matrix from the e basis to the f basis.

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 2 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 & -1 \end{array}\right)$$

$$\boxed{\underset{f\leftarrow e}{C} = \begin{pmatrix} -1 & 2 & 1 \\ 0 & 0 & 1 \\ 1 & -1 & -1 \end{pmatrix}}$$

d. Find the components of $p = 2 + 3x + 4x^2$ relative to the f basis.

$$p_f = \underset{f\leftarrow e}{C}\, p_e = \begin{pmatrix} -1 & 2 & 1 \\ 0 & 0 & 1 \\ 1 & -1 & -1 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} -2+6+4 \\ 0+0+4 \\ 2-3-4 \end{pmatrix}$$

$$\boxed{p_f = \begin{pmatrix} 8 \\ 4 \\ -5 \end{pmatrix}}$$

e. Find the polynomial q whose components relative to the f basis are $q_f = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$

Simplify fully.

$$q = 3f_1 + 2f_2 + 1f_3 = 3(1 + x) + 2(1 + x^2) + 1(2 + x)$$

$$\boxed{q = 7 + 4x + 2x^2}$$

2. (32 points) Let $P_2 = \{\text{polynomials of degree } < 2\}$ and $P_3 = \{\text{polynomials of degree } < 3\}$.
Consider the linear map $L : P_2 \to P_3$ given by $L(p) = 2\int_1^x p\,dx$.
For example: $L(3+4x) = 2\int_1^x (3+4x)\,dx = 2[3x+2x^2]_1^x = 2(3x+2x^2-5) = -10+6x+4x^2$.

a. Find the image of L. What is its dimension?
HINT: Take the general element of P_2 to be $p = a + bx$.

$$L(p) = 2\int_1^x (a+bx)\,dx = 2\left[ax + b\frac{x^2}{2}\right]_1^x = 2ax + bx^2 - 2a - b$$

$$\operatorname{Im}(L) = \{L(p)\} = \{2ax + bx^2 - 2a - b\}$$
$$= \{a(2x-2) + b(x^2-1)\} = Span(2x-2, x^2-1)$$

$\operatorname{Im}(L) = Span(2x-2, x^2-1)$

$\dim \operatorname{Im}(L) = 2$

b. Find the kernel of L. What is its dimension?

$$Ker(L) = \{p : L(p) = 2ax + bx^2 - 2a - b = 0\}$$
$$\Rightarrow \quad 2a = 0 \quad b = 0 \quad -2a - b = 0$$
$$\Rightarrow \quad a = b = 0 \quad \Rightarrow \quad p = 0$$

$Ker(L) = \{0\}$

$\dim Ker(L) = 0$

c. Is L onto? Why?

Because $\operatorname{Im}(L) = Span(2x-2, x^2-1) \neq Co-Dom(L) = P_3$
Equivalently, because $\dim \operatorname{Im}(L) = 2 \neq \dim Co-Dom(L) = 3$

Circle one:
Yes [✓ No]

d. Is L one-to-one? Why?

Because $Ker(L) = \{0\}$

Circle one:
[✓ Yes] No

e. Find the matrix of L relative to the standard bases.

$e_1 = 1 \qquad e_2 = x \qquad\qquad$ for P_2
$E_1 = 1 \qquad E_2 = x \qquad E_3 = x^2 \qquad$ for P_3

$$L(e_1) = L(1) = 2\int_1^x (1)\,dx = \Big[2x\Big]_1^x = 2x - 2 = -2E_1 + 2E_2$$
$$L(e_2) = L(x) = 2\int_1^x (x)\,dx = \Big[x^2\Big]_1^x = x^2 - 1 = -1E_1 + E_3$$

$$\underset{E\leftarrow e}{A} = \begin{pmatrix} -2 & -1 \\ 2 & 0 \\ 0 & 1 \end{pmatrix}$$

f. Find the null space of A. What is its dimension?

Solve $A\vec{x} = \vec{0}$: $\left(\begin{array}{cc|c} -2 & -1 & 0 \\ -2 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right) \Rightarrow \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow \vec{x} = \vec{0}$

$Null(A) = \{\vec{0}\}$

$\dim Null(A) = 0$

Equivalently, $Ker(L) = \{0\} \Rightarrow Null(A) = \{\vec{0}\}$

g. Find the column space of A. What is its dimension?

It is spanned by the columns of A.

$$\boxed{Col(A) = Span\left(\begin{pmatrix} -2 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}\right)}$$

$$\boxed{\dim Col(A) = 2}$$

h. Find the row space of A. What is its dimension?

It is spanned by the rows of A.

$Row(A) = Span((-2,-1),(2,0),(0,1))$

but $(-2,-1) = -(2,0) - (0,1)$

$$\boxed{Row(A) = Span((2,0),(0,1))}$$

$$\boxed{\dim Row(A) = 2}$$

3. $\big($15 points$\big)$ Consider the polynomial vector space $V = Span(x,x^2)$ with the inner product

$$\langle f,g\rangle = \int_0^1 \frac{fg}{x}\,dx$$

a. Find the angle between $v_1 = x$ and $v_2 = x^2$.

$$\langle v_1,v_2\rangle = \int_0^1 \frac{xx^2}{x}\,d\theta = \int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$$

$$\langle v_1,v_1\rangle = \int_0^1 \frac{(x)^2}{x}\,dx = \int_0^1 x\,dx = \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2} \qquad |v_1| = \frac{1}{\sqrt{2}}$$

$$\langle v_2,v_2\rangle = \int_0^1 \frac{(x^2)^2}{x}\,dx = \int_0^1 x^3\,dx = \left[\frac{x^4}{4}\right]_0^1 = \frac{1}{4} \qquad |v_2| = \frac{1}{2}$$

$$\cos\theta = \frac{\langle v_1,v_2\rangle}{|v_1||v_2|} = \frac{\frac{1}{3}}{\frac{1}{\sqrt{2}}\frac{1}{2}} = \frac{2\sqrt{2}}{3}$$

$$\boxed{\theta = \arccos\frac{2\sqrt{2}}{3}}$$

b. Start with the basis $v_1 = x$ and $v_2 = x^2$ and use the Gram-Schmidt procedure to produce an orthogonal basis w_1 and w_2 and an orthonormal basis u_1 and u_2.

$$w_1 = v_1 \qquad \boxed{w_1 = x}$$

$$\langle w_1,w_1\rangle = \langle v_1,v_1\rangle = \frac{1}{2} \qquad |w_1| = \frac{1}{\sqrt{2}}$$

$$u_1 = \frac{w_1}{|w_1|} \qquad \boxed{u_1 = \sqrt{2}\,x}$$

$$\langle v_2,w_1\rangle = \langle v_2,v_1\rangle = \frac{1}{3}$$

$$w_2 = v_2 - \frac{\langle v_2,w_1\rangle}{\langle w_1,w_1\rangle}w_1 = x^2 - \frac{\frac{1}{3}}{\frac{1}{2}}x \qquad \boxed{w_2 = x^2 - \frac{2}{3}x}$$

$$\langle w_2,w_2\rangle = \int_0^1 \frac{\left(x^2 - \frac{2}{3}x\right)^2}{x}\,dx = \int_0^1 \frac{x^4 - \frac{4}{3}x^3 + \frac{4}{9}x^2}{x}\,dx = \int_0^1\left(x^3 - \frac{4}{3}x^2 + \frac{4}{9}x\right)dx$$

$$= \left[\frac{x^4}{4} - \frac{4}{3}\frac{x^3}{3} + \frac{4}{9}\frac{x^2}{2}\right]_0^1 = \frac{1}{4} - \frac{4}{9} + \frac{2}{9} = \frac{9-8}{36} = \frac{1}{36} \qquad |w_2| = \frac{1}{6}$$

$$u_2 = \frac{w_2}{|w_2|} = 6\left(x^2 - \frac{2}{3}x\right) \qquad \boxed{u_2 = 6x^2 - 4x}$$

4. (35 points) Consider the matrix $A = \begin{pmatrix} 4 & 2 \\ -1 & 1 \end{pmatrix}$.

a. Find the eigenvalues of A. List them in ascending order.

$$|A - \lambda\mathbf{1}| = \begin{vmatrix} 4-\lambda & 2 \\ -1 & 1-\lambda \end{vmatrix} = (4-\lambda)(1-\lambda) + 2 = \lambda^2 - 5\lambda + 6 = (\lambda-2)(\lambda-3) = 0$$

$$\boxed{\lambda_1 = 2 \qquad \lambda_2 = 3}$$

b. Find the eigenvectors of A.

$$\boxed{\lambda_1 = 2:} \quad \left(\begin{array}{cc|c} 2 & 2 & 0 \\ -1 & -1 & 0 \end{array}\right) \quad \Rightarrow \quad \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \quad \Rightarrow \quad a + b = 0$$

$$\Rightarrow \quad \vec{x} = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -r \\ r \end{pmatrix} = r\begin{pmatrix} -1 \\ 1 \end{pmatrix} \qquad \boxed{e_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}}$$

$$\boxed{\lambda_2 = 3:} \quad \left(\begin{array}{cc|c} 1 & 2 & 0 \\ -1 & -2 & 0 \end{array}\right) \quad \Rightarrow \quad \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right) \quad \Rightarrow \quad a + 2b = 0$$

$$\Rightarrow \quad \vec{x} = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -2r \\ r \end{pmatrix} = r\begin{pmatrix} -2 \\ 1 \end{pmatrix} \qquad \boxed{e_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}}$$

c. Find a diagonal matrix D and an invertible matrix X so that $A = XDX^{-1}$.

A is a matrix relative to standard basis $\hat\imath$.

D is a matrix relative to the eigenbasis e whose diagonal entries are the eigenvalues.

$\underset{\hat\imath\leftarrow\hat\imath}{A} = \underset{\hat\imath\leftarrow e}{C}\ \underset{e\leftarrow e}{D}\ \underset{e\leftarrow\hat\imath}{C} = XDX^{-1}$ So $X = \underset{\hat\imath\leftarrow e}{C}$ which is the matrix whose columns are the eigenvectors.

$$\boxed{\underset{e\leftarrow e}{D} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}}$$

$$\boxed{X = \begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix}}$$

d. Find X^{-1}.

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \qquad X^{-1} = \frac{1}{-1+2}\begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix} \qquad \boxed{X^{-1} = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}}$$

e. Compute $\cos(\pi A)$.

HINT: If $D = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$, then $\pi D = \begin{pmatrix} \alpha\pi & 0 \\ 0 & \beta\pi \end{pmatrix}$. What is $\cos(\pi D)$?

$$\cos(\pi D) = \begin{pmatrix} \cos(2\pi) & 0 \\ 0 & \cos(3\pi) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\cos(\pi A) = \cos(X\pi DX^{-1}) = X\cos(\pi D)X^{-1} = \begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} -1-2 & -2-2 \\ 1+1 & 2+1 \end{pmatrix} \qquad \boxed{\cos(\pi A) = \begin{pmatrix} -3 & -4 \\ 2 & 3 \end{pmatrix}}$$