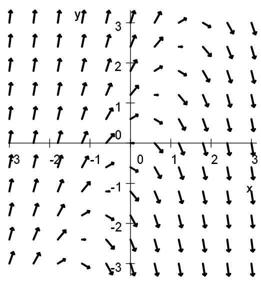
MATH 308 Exam I Solutions Summer 2003

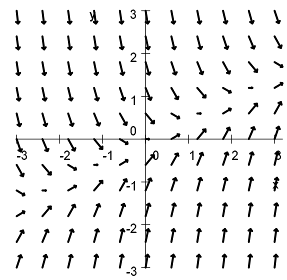
Section 301-302 Version B P. Yasskin

1. (5 points) Which of the following is the direction field of the differential equation $\frac{dy}{dx} = x - 2y$?

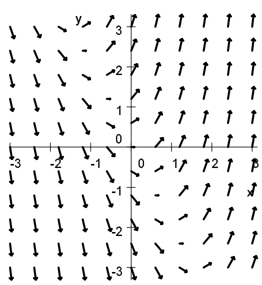
a



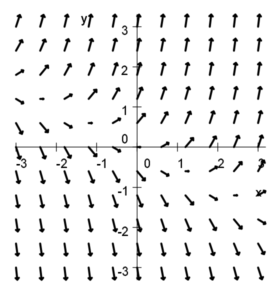
b



c



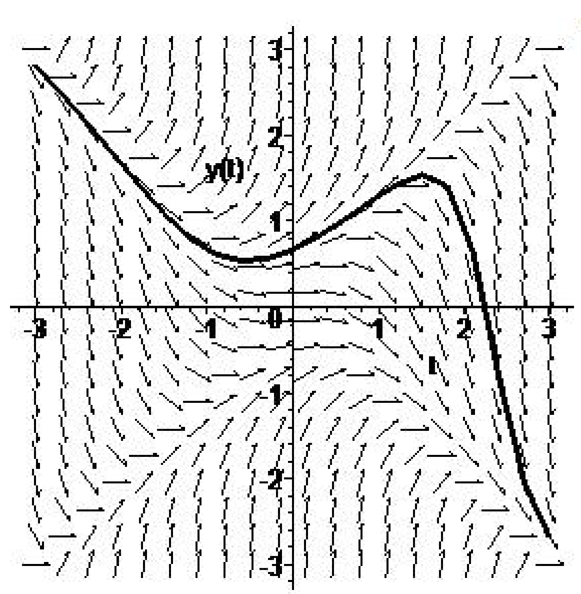
d



The slope is zero along the line $x - 2y = 0$ or $y = x/2$. In the second quadrant where $x < 0$ and $y > 0$, the slope is negative. These only occurs in plot (b).

2. (5 points) At the right is the direction field for a differential equation $\frac{dy}{dt} = F(t,y)$. Draw the solution curve which satisfies the initial condition $y(2) = 1$.

The curve must go through $(2,1)$ and be tangent to the vectors.



3. (10 points) Salt water is being added to a bucket of salt water with a different concentration, kept well mixed and emptied at the same rate. The amount of salt $S(t)$ in the bucket at time t satisfies the differential equation $\frac{dS}{dt} + 3S = 12.$

a. Draw the phase line diagram for this differential equation.

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$$\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow | \leftarrow\leftarrow\leftarrow\leftarrow\leftarrow\leftarrow\leftarrow\leftarrow\leftarrow\leftarrow\leftarrow \; S$$

b. If the initial quantity of salt is $S(0) = 2$, find the asymptotic quantity of salt.

$$\lim_{t\to\infty} S(t) = 4$$

c. If the initial quantity of salt is $S(0) = 7$, find the asymptotic quantity of salt.

$$\lim_{t\to\infty} S(t) = 4$$

4. (20 points) Use Euler's method to approximate the solution to the initial value problem $\frac{dy}{dx} = \frac{x}{y}$ with $y(1) = 2$. Take the step size to be $h = 0.2$ and compute 2 steps. Thus you need to find (x_0,y_0), (x_1,y_1), (x_2,y_2).

$$F(x,y) = \frac{x}{y}$$

$$x_0 = 1 \qquad y_0 = 2$$

$$x_1 = x_0 + h = 1.2 \qquad y_1 = y_0 + \frac{x_0}{y_0}h = 2 + \frac{1}{2}.2 = 2.1$$

$$x_2 = x_1 + h = 1.4 \qquad y_2 = y_1 + \frac{x_1}{y_1}h = 2.1 + \frac{1.2}{2.1}.2 = 2.2143$$

5. (10 points) Solve the initial value problem: $\frac{dy}{dx} = \frac{x}{y} + xy$ with $y(0) = -1$

Factor RHS: $\frac{dy}{dx} = \frac{x(1+y^2)}{y}$

Separate: $\int \frac{y}{1+y^2}\, dy = \int x\, dx$

Integrate: $\frac{1}{2}\ln|1+y^2| = \frac{1}{2}x^2 + C$

Solve for y: $\ln|1+y^2| = x^2 + 2C$ $\quad |1+y^2| = e^{2C}e^{x^2}$ $\quad 1+y^2 = \pm e^{2C}e^{x^2} = Ae^{x^2}$

$y = \pm\sqrt{Ae^{x^2} - 1}$

Initial condition: $x = 0,$ $\quad y = -1:$ $\quad -1 = \pm\sqrt{A-1}$ $\quad A = 2$ and need – sign.

Solution: $y = -\sqrt{2e^{x^2} - 1}$

6. (10 points) Solve the initial value problem: $\frac{dy}{dx} = 2y + \frac{1}{y}e^{2x}$ with $y(0) = 2$

using the change of variables $z = y^2$.

Change of Variables: $\frac{dz}{dx} = 2y\frac{dy}{dx} = 2y\left(2y + \frac{1}{y}e^{2x}\right) = 4y^2 + 2e^{2x} = 4z + 2e^{2x}$

Standard Linear Form: $\frac{dz}{dx} - 4z = 2e^{2x}$

Identify P and integrate: $P = -4$ $\quad \int P\, dx = -4x$

Integrating Factor: $I = \exp\left(\int P\, dx\right) = e^{-4x}$

Multiply Std Form by I: $e^{-4x}\frac{dz}{dx} - 4e^{-4x}z = 2e^{-2x}$ or $\frac{d}{dx}(e^{-4x}z) = 2e^{-2x}$

Integrate: $e^{-4x}z = -e^{-2x} + C$

Solve for z: $z = -e^{2x} + Ce^{4x}$

Initial condition: $x = 0,$ $\quad y = 2,$ $\quad z = 4:$ $\quad 4 = -1 + C$ $\quad C = 5$

Solution: $z = -e^{2x} + 5e^{4x}$

Substitute back: $y = \sqrt{z} = \sqrt{5e^{4x} - e^{2x}}$

7. (10 points) Solve the initial value problem: $x^3 \dfrac{dy}{dx} = 5x^2 y + 6x^4$ with $y(1) = 2$

Standard Linear Form: $\dfrac{dy}{dx} - \dfrac{5}{x}y = 6x$

Identify P and integrate: $P = -\dfrac{5}{x}$ $\quad \int P\,dx = -5\ln x = \ln(x^{-5})$

Integrating Factor: $I = \exp\left(\int P\,dx\right) = \exp(\ln(x^{-5})) = x^{-5}$

Multiply Std Form by I: $x^{-5}\dfrac{dy}{dx} - 5x^{-6}y = 6x^{-4}$ or $\dfrac{d}{dx}(x^{-5}y) = 6x^{-4}$

Integrate: $x^{-5}y = -2x^{-3} + C$

Solve for y: $y = -2x^2 + Cx^5$

Initial condition: $x = 1,$ $y = 2$: $2 = -2 + C$ $\quad C = 4$

Solution: $y = -2x^2 + 4x^5$

8. (10 points) Solve the initial value problem: $\dfrac{dy}{dx} = 1 + 2\dfrac{y}{x}$ with $y(2) = 6$

using the change of variables $y = xz$.

HINT: On the LHS use the product rule. On the RHS just substitute.
(This substitution works whenever the RHS is a function of $\frac{y}{x}$.)

LHS: $\dfrac{dy}{dx} = x\dfrac{dz}{dx} + z$ $\qquad$ RHS: $1 + 2\dfrac{y}{x} = 1 + 2z$

Equate: $x\dfrac{dz}{dx} + z = 1 + 2z$ $\qquad$ Simplify: $x\dfrac{dz}{dx} = 1 + z$

Separate: $\displaystyle\int \frac{1}{1+z}\,dz = \int \frac{1}{x}\,dx$

Integrate: $\ln|1+z| = \ln|x| + C$

Solve: $|1+z| = e^C|x|$ $\qquad 1 + z = \pm e^C x = Ax$ $\qquad z = Ax - 1$

Substitute back: $y = xz = Ax^2 - x$

Initial condition: $x = 2,$ $y = 6$: $6 = 4A - 2$ $\quad A = 2$

Substitute back: $y = 2x^2 - x$

9. (10 points) For the following problem, define your variables and set up the differential equation and initial condition. Do not solve the equations.

A swimming pool contains $8,000$ gallons of water with 0.01% chlorine. Starting at $2:00$ PM, city water containing 0.002% chlorine is pumped into the pool at 4 gallons per minute. The pool water flows out at the same rate. What is the percentage of chlorine in the pool at $3:00$ PM?

Let $y(t)$ be the gallons of chlorine in the pool at time t in minutes, where $t = 0$ at $2:00$ PM.

$$\frac{dy}{dt} = 4\frac{\text{gal } H_20}{\text{min}} \times \frac{0.002 \text{ gal Cl}}{100 \text{ gal } H_20} - 4\frac{\text{gal } H_20}{\text{min}} \times \frac{y(t) \text{ gal Cl}}{8,000 \text{ gal } H_20}$$

Equation: $\frac{dy}{dt} = 8 \times 10^{-5} - 5 \times 10^{-4}y$

Initial Condition: $y(0) = 8,000 \text{ gal } H_20 \times \frac{0.01 \text{ gal Cl}}{100 \text{ gal } H_20} = 0.8 \text{ gal Cl}$

10. (10 points) For the following problem, define your variables, set up the differential equation and initial condition and identify the equation you solve to determine the unknown constants. Do not solve the equations.

A pot of water is put on the stove to boil. Initially the pot and water are at $20°$C. When the stove is turned on, the pot heats up so that its temperature increases according to the formula $P(t) = \frac{20 + 120t}{1 + t}$ where t is measured in minutes. Thus $P(0) = 20°$C and $\lim_{t \to \infty} = 120°$C. After 1 minute the water has reached $40°$C. Assuming Newton's Law of Heating, how long does it take until the water reaches $100°$C and starts boiling?

Let $T(t)$ be the temperature of the water at time t.

Equation: $\frac{dT}{dt} = k(P(t) - T) = k\left(\frac{20 + 120t}{1 + t} - T\right)$

Initial Condition: $T(0) = 20$ to determine the constant of integration.

Second Condition: $T(1) = 40$ to determine the constant k.