MATH 308 Exam 2 Solutions Summer 2003

Section 301-302 Version A P. Yasskin

1. (20 points) Consider the homogenous differential equation $\dfrac{d^2x}{dt^2} + 7\dfrac{dx}{dt} + 12x = 0$

a. (10) Find the general solution.

$r^2 + 7r + 12 = 0 \qquad (r+3)(r+4) = 0 \qquad r = -3, -4 \qquad x(t) = Ae^{-3t} + Be^{-4t}$

b. (8) Find the specific solution satisfying the initial conditions $x(0) = 0, \quad x'(0) = -2.$

$$\begin{array}{llllllll} x(t) = Ae^{-3t} + Be^{-4t} & \Rightarrow & x(0) = A + B = 0 & \Rightarrow & B = -A & \Rightarrow & A = -2 \\ x'(t) = -3Ae^{-3t} - 4Be^{-4t} & & x'(0) = -3A - 4B = -2 & & A = -2 & & B = 2 \end{array}$$

Therefore: $x(t) = -2e^{-3t} + 2e^{-4t}$

c. (2) If you regard this equation as describing a free, damped harmonic oscillator, it is

Circle one: i) underdamped ii) critically damped [iii) overdamped]

2. (30 points) Consider the inhomogenous differential equation $\dfrac{d^2x}{dt^2} + 7\dfrac{dx}{dt} + 12x = 150\cos(3t)$

HINT: The related homogenous differential equation was analyzed in problem 1.

a. (10) Find a particular solution.

Guess: $x = P\cos(3t) + Q\sin(3t)$

Then: $x' = -3P\sin(3t) + 3Q\cos(3t) \qquad x'' = -9P\cos(3t) - 9Q\sin(3t)$

$[-9P\cos(3t) - 9Q\sin(3t)] + 7[-3P\sin(3t) + 3Q\cos(3t)] + 12[P\cos(3t) + Q\sin(3t)] = 150\cos(3t)$

$(-9P + 21Q + 12P)\cos(3t) + (-9Q - 21P + 12Q)\sin(3t) = 150\cos(3t)$

$$\begin{array}{lllll} 3P + 21Q = 150 & \Rightarrow & Q = 7P & \Rightarrow & P = 1 \\ -21P + 3Q = 0 & & 3P + 147P = 150 & & Q = 7 \end{array}$$

Therefore: $x = \cos(3t) + 7\sin(3t)$

b. (5) Find the general solution. (Use your result from 1a.)

$x(t) = Ae^{-3t} + Be^{-4t} + \cos(3t) + 7\sin(3t)$

c. (10) Find the specific solution satisfying the initial conditions $x(0) = 5, \quad x'(0) = 5.$

$$\begin{array}{lll} x(t) = Ae^{-3t} + Be^{-4t} + \cos(3t) + 7\sin(3t) & \Rightarrow & x(0) = A + B + 1 = 5 \\ x'(t) = -3Ae^{-3t} - 4Be^{-4t} - 3\sin(3t) + 21\cos(3t) & & x'(0) = -3A - 4B + 21 = 5 \end{array}$$

$$\begin{array}{llll} \Rightarrow & B = 4 - A & \Rightarrow & A = 0 \\ & A + 5 = 5 & & B = 4 \end{array}$$

Therefore: $x(t) = 4e^{-4t} + \cos(3t) + 7\sin(3t)$

d. (5) What is the phase shift? What is the gain?

HINT: Write the steady state solution as $A\cos(3t - \varphi)$

$\cos(3t) + 7\sin(3t) = A\cos(3t - \varphi) = A\cos(3t)\cos\varphi + A\sin(3t)\sin\varphi$

$$\begin{array}{lll} A\cos\varphi = 1 & \Rightarrow & A = \sqrt{50} = 5\sqrt{2} \\ A\sin\varphi = 7 & & \varphi = \tan^{-1}7 \end{array}$$

The phase shift is $\varphi = \tan^{-1}7 \approx 1.4289.$ The gain is $\dfrac{A}{150} = \dfrac{5\sqrt{2}}{150} = \dfrac{\sqrt{2}}{30} \approx 0.04714$

3. (10 points) Consider the inhomogenous differential equation $\frac{d^2x}{dt^2} + 7\frac{dx}{dt} + 12x = 5e^{-4t}$

Find a particular solution.

HINT: The related homogenous differential equation was analyzed in problem 1.

We cannot guess $x = Pe^{-4t}$ because e^{-4t} is a solution of the homogeneous equation.

Guess: $x = Pte^{-4t}$

Then: $x' = Pe^{-4t} - 4Pte^{-4t}$ $\quad x'' = -4Pe^{-4t} - 4Pe^{-4t} + 16Pte^{-4t}$

$[-4Pe^{-4t} - 4Pe^{-4t} + 16Pte^{-4t}] + 7[Pe^{-4t} - 4Pte^{-4t}] + 12[Pte^{-4t}] = 5e^{-4t}$

$(16P - 28P + 12P)te^{-4t} + (-8P + 7P)e^{-4t} = 5e^{-4t}$

$(-P)e^{-4t} = 5e^{-4t} \quad \Rightarrow \quad P = -5$

Therefore: $x = -5te^{-4t}$

4. (10 points) Consider the homogenous differential equation $\frac{d^2x}{dt^2} + 8\frac{dx}{dt} + 16x = 0$

a. (8) Find the general solution.

$r^2 + 8r + 16 = 0 \qquad r = \frac{-8 \pm \sqrt{64 - 64}}{2} = -4$ (double root) $\qquad x(t) = Ae^{-4t} + Bte^{-4t}$

b. (2) If you regard this equation as describing a free, damped harmonic oscillator, it is

Circle one: i) underdamped **[ii) critically damped]** iii) overdamped

5. (10 points) Consider the homogenous differential equation $\frac{d^2x}{dt^2} + 8\frac{dx}{dt} + 25x = 0$

a. (8) Find the general solution.

$r^2 + 8r + 25 = 0 \qquad r = \frac{-8 \pm \sqrt{64 - 100}}{2} = -4 \pm 3i \qquad x(t) = Ae^{-4t}\cos(3t) + Be^{-4t}\sin(3t)$

b. (2) If you regard this equation as describing a free, damped harmonic oscillator, it is

Circle one: **[i) underdamped]** ii) critically damped iii) overdamped

6. (10 points) Consider the inhomogenous differential equation $\frac{d^2x}{dt^2} + 8\frac{dx}{dt} + 25x = 50t^2 + 32t$

Find a particular solution.

HINT: The related homogenous differential equation was analyzed in problem 5.

Guess: $x = At^2 + Bt + C$

Then: $x' = 2At + B \qquad x'' = 2A$

$(2A) + 8(2At + B) + 25(At^2 + Bt + C) = 50t^2 + 32t$

$(25A)t^2 + (16A + 25B)t + (2A + 8B + 25C) = 50t^2 + 32t$

$$\begin{aligned} 25A &= 50 \\ 16A + 25B &= 32 \\ 2A + 8B + 25C &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} A &= 2 \\ B &= 0 \\ C &= -\frac{4}{25} \end{aligned}$$

Therefore: $x = 2t^2 - \frac{4}{25}$

7. (10 points) A 2 kg mass is attached to a spring with spring constant of 3 N/m and feels air resistance proportional to the velocity with drag coefficient of 0.5 N-sec/m. The mass also has an electric charge and there is an electric field which applies an external force of $F_e = 4\cos(7t)$. If you hold the mass at .6 m from its rest position and let go at $t = 0$, write the differential equation and initial conditions which determine the motion of the mass.

$$2\frac{d^2x}{dt^2} + 0.5\frac{dx}{dt} + 3x = 4\cos(7t) \qquad x(0) = .6 \qquad x'(0) = 0$$