MATH 308 | Exam 2 Solutions | Summer 2003
Section 301-302 | Version B | P. Yasskin

1. (20 points) Consider the homogenous differential equation $\frac{d^2x}{dt^2} + 7\frac{dx}{dt} + 10x = 0$

a. (10) Find the general solution.

$r^2 + 7r + 10 = 0 \qquad (r+2)(r+5) = 0 \qquad r = -2, -5 \qquad x(t) = Ae^{-2t} + Be^{-5t}$

b. (8) Find the specific solution satisfying the initial conditions $x(0) = 3, \quad x'(0) = 0.$

$$\begin{array}{l} x(t) = Ae^{-2t} + Be^{-5t} \\ x'(t) = -2Ae^{-2t} - 5Be^{-5t} \end{array} \Rightarrow \begin{array}{l} x(0) = A + B = 3 \\ x'(0) = -2A - 5B = 0 \end{array} \Rightarrow \begin{array}{l} A = -\frac{5}{2}B \\ -\frac{5}{2}B + B = 3 \end{array} \Rightarrow \begin{array}{l} B = -2 \\ A = 5 \end{array}$$

Therefore: $x(t) = 5e^{-2t} - 2e^{-5t}$

c. (2) If you regard this equation as describing a free, damped harmonic oscillator, it is

Circle one: i) underdamped ii) critically damped **[iii) overdamped]**

2. (30 points) Consider the inhomogenous differential equation $\frac{d^2x}{dt^2} + 7\frac{dx}{dt} + 10x = 442\cos(3t)$

HINT: The related homogenous differential equation was analyzed in problem 1.

a. (10) Find a particular solution.

Guess: $x = P\cos(3t) + Q\sin(3t)$

Then: $x' = -3P\sin(3t) + 3Q\cos(3t) \qquad x'' = -9P\cos(3t) - 9Q\sin(3t)$

$[-9P\cos(3t) - 9Q\sin(3t)] + 7[-3P\sin(3t) + 3Q\cos(3t)] + 10[P\cos(3t) + Q\sin(3t)] = 442\cos(2t)$

$(-9P + 21Q + 10P)\cos(3t) + (-9Q - 21P + 10Q)\sin(3t) = 442\cos(3t)$

$$\begin{array}{l} P + 21Q = 442 \\ -21P + Q = 0 \end{array} \Rightarrow \begin{array}{l} Q = 21P \\ P + 21^2 = 442 \end{array} \Rightarrow \begin{array}{l} P = 1 \\ Q = 21 \end{array}$$

Therefore: $x = \cos(3t) + 21\sin(3t)$

b. (5) Find the general solution. (Use your result from 1a.)

$x(t) = Ae^{-2t} + Be^{-5t} + \cos(3t) + 21\sin(3t)$

c. (10) Find the specific solution satisfying the initial conditions $x(0) = 1, \quad x'(0) = 0.$

$$\begin{array}{l} x(t) = Ae^{-2t} + Be^{-5t} + \cos(3t) + 21\sin(3t) \\ x'(t) = -2Ae^{-2t} - 5Be^{-5t} - 3\sin(3t) + 63\cos(3t) \end{array} \Rightarrow \begin{array}{l} x(0) = A + B + 1 = 1 \\ x'(0) = -2A - 5B + 63 = 0 \end{array}$$

$$\Rightarrow \begin{array}{l} B = -A \\ 3A + 63 = 0 \end{array} \Rightarrow \begin{array}{l} A = -21 \\ B = 21 \end{array}$$

Therefore: $x(t) = -21e^{-2t} + 21e^{-5t} + \cos(3t) + 21\sin(3t)$

d. (5) What is the phase shift? What is the gain?

HINT: Write the steady state solution as $A\cos(3t - \varphi)$

$\cos(3t) + 21\sin(3t) = A\cos(3t - \varphi) = A\cos(3t)\cos\varphi + A\sin(3t)\sin\varphi$

$$\begin{array}{l} A\cos\varphi = 1 \\ A\sin\varphi = 21 \end{array} \Rightarrow \begin{array}{l} A = \sqrt{442} \approx 21.024 \\ \varphi = \tan^{-1}21 \end{array}$$

The phase shift is $\varphi = \tan^{-1}21 \approx 1.5232$. The gain is $\frac{A}{442} = \frac{\sqrt{442}}{442} \approx 0.047565$

3. (10 points) Consider the inhomogenous differential equation $\frac{d^2x}{dt^2} + 7\frac{dx}{dt} + 10x = 6e^{-2t}$

Find a particular solution.

HINT: The related homogenous differential equation was analyzed in problem 1.

We cannot guess $x = Pe^{-2t}$ because e^{-2t} is a solution of the homogeneous equation.

Guess: $x = Pte^{-2t}$

Then: $x' = Pe^{-2t} - 2Pte^{-2t}$ $\quad x'' = -2Pe^{-2t} - 2Pe^{-2t} + 4Pte^{-2t}$

$[-2Pe^{-2t} - 2Pe^{-2t} + 4Pte^{-2t}] + 7[Pe^{-2t} - 2Pte^{-2t}] + 10[Pte^{-2t}] = 6e^{-2t}$

$(4P - 14P + 10P)te^{-2t} + (-4P + 7P)e^{-2t} = 6e^{-2t}$

$(3P)e^{-2t} = 6e^{-2t} \quad \Rightarrow \quad P = 2$

Therefore: $x = 2te^{-2t}$

4. (10 points) Consider the homogenous differential equation $\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 25x = 0$

a. (8) Find the general solution.

$r^2 + 6r + 25 = 0 \qquad r = \frac{-6 \pm \sqrt{36 - 100}}{2} = -3 \pm 4i \qquad x(t) = Ae^{-3t}\cos(4t) + Be^{-3t}\sin(4t)$

b. (2) If you regard this equation as describing a free, damped harmonic oscillator, it is

Circle one: [i) underdamped] ii) critically damped iii) overdamped

5. (10 points) Consider the homogenous differential equation $\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 9x = 0$

a. (8) Find the general solution.

$r^2 + 6r + 9 = 0 \qquad r = \frac{-6 \pm \sqrt{36 - 36}}{2} = -3$ (double root) $\qquad x(t) = Ae^{-3t} + Bte^{-3t}$

b. (2) If you regard this equation as describing a free, damped harmonic oscillator, it is

Circle one: i) underdamped [ii) critically damped] iii) overdamped

6. (10 points) Consider the inhomogenous differential equation $\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 9x = 27t^2 - 18$

Find a particular solution.

HINT: The related homogenous differential equation was analyzed in problem 5.

Guess: $x = At^2 + Bt + C$

Then: $x' = 2At + B$ $\quad x'' = 2A$

$(2A) + 6(2At + B) + 9(At^2 + Bt + C) = 27t^2 - 18$

$(9A)t^2 + (12A + 9B)t + (2A + 6B + 9C) = 27t^2 - 18$

$$\begin{aligned} 9A &= 27 \\ 12A + 9B &= 0 \\ 2A + 6B + 9C &= -18 \end{aligned} \quad \Rightarrow \quad \begin{aligned} A &= 3 \\ B &= -4 \\ C &= 0 \end{aligned}$$

Therefore: $x = 3t^2 - 4t$

7. (10 points) A 3 kg mass is attached to a spring with spring constant of 6 N/m and feels air resistance proportional to the velocity with drag coefficient of 0.4 N-sec/m. The mass also has an electric charge and there is an electric field which applies an external force of $F_e = 5\cos(2t)$. If you hold the mass at .8 m from its rest position and let go at $t = 0$, write the differential equation and initial conditions which determine the motion of the mass.

$$3\frac{d^2x}{dt^2} + 0.4\frac{dx}{dt} + 6x = 5\cos(2t) \qquad x(0) = .8 \qquad x'(0) = 0$$