Name______________________ NetID________________

MATH 308	Final Exam	Spring 2009
Section 511	Solutions	P. Yasskin

1	/20	4	/10
2	/20	5	/20
3	/10	6	/20
Total			/100

1. (20 points) Consider the second order non-homogeneous differential equation

$$y'' + 5y' + 4y = 6e^{-x} + 8x^2 + 3.$$

a. Find two solutions of the homogeneous differential equation $y'' + 5y' + 4y = 0$. Verify they are linearly independent. Give the general homogeneous solution.

$y = e^{rx}$ $\quad r^2 + 5r + 4 = 0$ $\quad (r+1)(r+4) = 0$ $\quad r = -1, -4$

Two homogeneous solutions are $\boxed{y_1 = e^{-x} \qquad y_2 = e^{-4x}}$

Assume $ay_1 + by_2 = 0$ $\quad ae^{-x} + be^{-4x} = 0$ for all x.

$x = 0$: $a + b = 0$

$x = -1$: $ae + be^4 = 0$ $\quad \Rightarrow \quad b = -a$ and $a(e - e^4) = 0 \quad \Rightarrow \quad a = 0, b = 0$

OR $\quad W = \begin{vmatrix} e^{-x} & e^{-4x} \\ -e^{-x} & -4e^{-4x} \end{vmatrix} = -4e^{-5x} + e^{-5x} = -3e^{-5x} \neq 0$

So y_1 and y_2 are linearly independent.

The general homogemneous solution is $\boxed{y_h = c_1e^{-x} + c_2e^{-4x}.}$

b. Use undetermined coefficients to find a particular non-homogeneous solution.

$y_p = Axe^{-x} + Bx^2 + Cx + D$ $\qquad$ The x is needed on the first term because $y_1 = e^{-x}$.

$y_p' = Ae^{-x} - Axe^{-x} + 2Bx + C$ $\qquad y_p'' = -2Ae^{-x} + Axe^{-x} + 2B$

$y_p'' + 5y_p' + 4y_p = 6e^{-x} + 8x^2 + 3 \qquad \Rightarrow$

$(-2Ae^{-x} + Axe^{-x} + 2B) + 5(Ae^{-x} - Axe^{-x} + 2Bx + C) + 4(Axe^{-x} + Bx^2 + Cx + D) = 6e^{-x} + 8x^2 + 3$

$xe^{-x}(A - 5A + 4A) + e^{-x}(-2A + 5A) + x^2(4B) + x(10B + 4C) + 1(2B + 5C + 4D) = 6e^{-x} + 8x^2 + 3$

$e^{-x}(3A) + x^2(4B) + x(10B + 4C) + 1(2B + 5C + 4D) = 6e^{-x} + 8x^2 + 3$

$3A = 6 \qquad 4B = 8 \qquad 10B + 4C = 0 \qquad 2B + 5C + 4D = 3$

$A = 2 \qquad B = 2 \qquad C = -5 \qquad D = 6$

$\boxed{y_p = 2xe^{-x} + 2x^2 - 5x + 6}$

c. Find the non-homogeneous solution satisfying the initial conditions: $y(0) = 6 \quad y'(0) = 3.$

$y = y_h + y_p = c_1e^{-x} + c_2e^{-4x} + 2xe^{-x} + 2x^2 - 5x + 6 \qquad y(0) = c_1 + c_2 + 6 = 6$

$y' = -c_1e^{-x} - 4c_2e^{-4x} + 2e^{-x} - 2xe^{-x} + 4x - 5 \qquad y'(0) = -c_1 - 4c_2 + 2 - 5 = 3$

Add: $-3c_2 + 3 = 9 \qquad c_2 = -2 \qquad c_1 = -c_2 = 2$

$\boxed{y = 2e^{-x} - 2e^{-4x} + 2xe^{-x} + 2x^2 - 5x + 6}$

2. (20 points) Consider the second order homogeneous differential equation

$$x^2y'' + (-2x - 2x^2)y' + (2 + 2x + x^2)y = 0.$$

a. Verify $y_1 = xe^x$ is a solution. (Show your algebra!)

$y_1' = e^x + xe^x \qquad y_1'' = 2e^x + xe^x$

$x^2y_1'' + (-2x - 2x^2)y_1' + (2 + 2x + x^2)y_1 = x^2(2e^x + xe^x) + (-2x - 2x^2)(e^x + xe^x) + (2 + 2x + x^2)(xe^x$

$= [x^2(2 + x) + (-2x - 2x^2)(1 + x) + (2 + 2x + x^2)x]e^x$

$= [(2x^2 + x^3) + (-2x - 2x^2) + (-2x^2 - 2x^3) + (2x + 2x^2 + x^3)]e^x = 0$

b. Use reduction of order (similar to variation of parameters) to find a second solution. (Be careful with your algebra. Nearly everything should cancel.)

$y_2 = vxe^x \qquad y_2' = v'xe^x + v(e^x + xe^x) \qquad y_2'' = v''xe^x + 2v'(e^x + xe^x) + v(2e^x + xe^x)$

$x^2y_2'' + (-2x - 2x^2)y_2' + (2 + 2x + x^2)y_2 = 0$

$x^2[v''xe^x + 2v'(e^x + xe^x) + v(2e^x + xe^x)] + (-2x - 2x^2)[v'xe^x + v(e^x + xe^x)] + (2 + 2x + x^2)[vxe^x] =$

Divide by e^x and start expanding:

$v''x^3 + 2v'(1 + x)x^2 + v(2 + x)x^2 + v'x(-2x - 2x^2) + v(1 + x)(-2x - 2x^2) + vx(2 + 2x + x^2) = 0$

Group terms by power of v:

$v''[x^3] + v'[2(1 + x)x^2 + x(-2x - 2x^2)] + v[(2 + x)x^2 + (1 + x)(-2x - 2x^2) + x(2 + 2x + x^2)] = 0$

Expand:

$v''[x^3] + v'[0] + v[0] = 0$

So $v'' = 0 \qquad v' = C_1 \qquad v = C_1x + C_2$

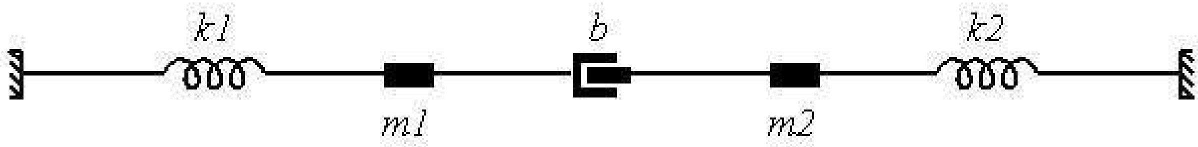
$y_2 = vxe^x = (C_1x + C_2)xe^x = C_1x^2e^x + C_2xe^x$

We set $C_2 = 0$ because xe^x reproduces the first solution.

We set $C_1 = 1$ because we only need one solution.

$\boxed{y_2 = x^2e^x}$

3. (10 points) Consider the mass-spring-piston system shown in the figure.



The masses are $m_1 = 6$ kg and $m_2 = 8$ kg.
The spring constants are $k_1 = 3$ N/m and $k_2 = 5$ N/m.
The piston pulls in with a force proportional to the velocity with which it is expanding or pushes out with a force proportional to the velocity with which it is contracting with a proportionality constant which is the drag coefficient of $b = 4$ N•sec/m.
Initially, mass m_1 is moved 2 m to the right and given a velocity of 5 m/sec to the left, while mass m_2 is moved 4 m to the left and given a velocity of 3 m/sec to the right.
Let $x(t)$ be the displacement of m_1 from it's rest position measured positive to the right.
Let $y(t)$ be the displacement of m_2 from it's rest position measured positive to the right.
Set up **second** order **differential equations** and **initial conditions** for x and y.
Do not solve the equations.

The motion of m_1 is given by:

$$m_1 x'' = -k_1 x + b(y' - x') \quad \text{or} \quad \boxed{6x'' = -3x + 4(y' - x')}$$

Here, the coefficient of $k_1 x$ is negative because when m_1 is to the right of its rest position, $(x > 0)$ the spring is stretched and pulls to the left.
Similarly, the coefficient of $b(y' - x')$ is positive because when m_2 is moving to the right faster than m_1, $(y' - x' > 0)$ the piston is expanding and the force on m_1 is to the right.

The motion of m_2 is given by:

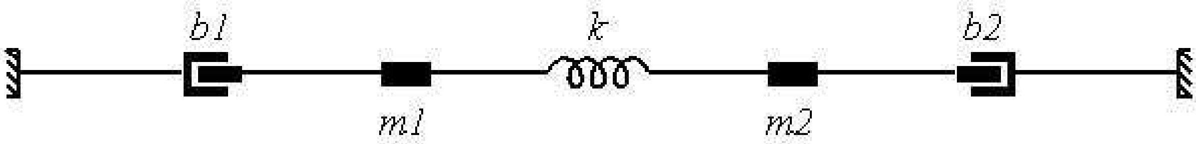
$$m_2 y'' = -k_2 y - b(y' - x') \quad \text{or} \quad \boxed{8y'' = -5y - 4(y' - x')}$$

Here, the coefficient of $k_2 y$ is negative because when m_2 is to the right of its rest position, $(y > 0)$ the spring is compressed and pushes to the left.
The coefficient of $b(y' - x')$ is negative because when m_2 is moving to the right faster than m_1, $(y' - x' > 0)$ the piston is expanding and the force on m_2 is to the left.

The initial conditions are:

$$\boxed{x(0) = 2 \qquad x'(0) = -5 \qquad y(0) = -4 \qquad y'(0) = 3}$$

4. (10 points) Consider the mass-spring-piston system shown in the figure.



The masses are $m_1 = 1$ kg and $m_2 = 1$ kg. The spring constants is $k = 7$ N/m.
The drag coefficient for the pistons are $b_1 = 5$ N•sec/m and $b_2 = 6$ N•sec/m.
Each piston pulls in with a force proportional to the velocity with which it is expanding or pushes out with a force proportional to the velocity with which it is contracting.
Initially, mass m_1 is moved 1 m to the right and given a velocity of 2 m/sec to the right, while mass m_2 is moved 3 m to the left and given a velocity of 4 m/sec to the left.
Let $x(t)$ be the displacement of m_1 from it's rest position measured positive to the right.
Let $y(t)$ be the displacement of m_2 from it's rest position measured positive to the right.
The second order differential equations and initial conditions for x and y are

$$m_1 x'' = -b_1 x' + k(y - x) \quad \text{or} \quad \boxed{x'' = -5x' + 7(y - x)} \qquad \boxed{x(0) = 1 \qquad x'(0) = 2}$$

$$m_2 y'' = -b_2 y' - k(y - x) \quad \text{or} \quad \boxed{y'' = -6y' - 7(y - x)} \qquad \boxed{y(0) = -3 \qquad y'(0) = -4}$$

Let $p = x'$ and $q = y'$.
Set up **first** order **differential equations** $\vec{x}' = A\vec{x}$ for $\vec{x} = \begin{pmatrix} x \\ p \\ y \\ q \end{pmatrix}$.
and **initial conditions** for $\vec{x}(0)$.
Do not solve the equations.

$$\vec{x}' = \begin{pmatrix} x' \\ p' \\ y' \\ q' \end{pmatrix} = \begin{pmatrix} p \\ x'' \\ q \\ y'' \end{pmatrix} = \begin{pmatrix} p \\ -5x' + 7(y - x) \\ q \\ -6y' - 7(y - x) \end{pmatrix} = \begin{pmatrix} p \\ -5p + 7(y - x) \\ q \\ -6q - 7(y - x) \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 0 & 0 \\ -7 & -5 & 7 & 0 \\ 0 & 0 & 0 & 1 \\ 7 & 0 & -7 & -6 \end{pmatrix} \begin{pmatrix} x \\ p \\ y \\ q \end{pmatrix} = A\vec{x} \quad \text{where} \quad \boxed{A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -7 & -5 & 7 & 0 \\ 0 & 0 & 0 & 1 \\ 7 & 0 & -7 & -6 \end{pmatrix}}$$

$$\boxed{\vec{x}(0) = \begin{pmatrix} x(0) \\ p(0) \\ y(0) \\ q(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -3 \\ -4 \end{pmatrix}}$$

5. (20 points) Consider the first order differential equations $\vec{x}' = A\vec{x}$

where $\vec{x} = \begin{pmatrix} x \\ p \\ y \\ q \end{pmatrix}$ and $A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & -3 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & -3 \end{pmatrix}$.

Find the **eigenvalues** and **eigenvectors** of A.

Then find the **general solution** of the differential equation. (Vector form is OK.)

HINT: Two of the eigenvalues and eigenvectors are:

$r = -2 \qquad \vec{u}_{-2} = \begin{pmatrix} 1 \\ -2 \\ -1 \\ 2 \end{pmatrix} \qquad\qquad r = -3 \qquad \vec{u}_{-3} = \begin{pmatrix} -1 \\ 3 \\ -1 \\ 3 \end{pmatrix}$

$$\det(A - r\mathbf{1}) = \begin{vmatrix} -r & 1 & 0 & 0 \\ -1 & -3-r & 1 & 0 \\ 0 & 0 & -r & 1 \\ 1 & 0 & -1 & -3-r \end{vmatrix} = r^4 + 6r^3 + 11r^2 + 6r = r(r+1)(r+2)(r+3)$$

$r = 0$:

$$\left(\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 0 \\ -1 & -3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & -3 & 0 \end{array}\right) \Rightarrow \left(\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -0 & 0 \end{array}\right) \Rightarrow \vec{u}_0 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$r = -1$:

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ -1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & -1 & -2 & 0 \end{array}\right) \Rightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \vec{u}_{-1} = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$$

$$\boxed{\vec{x} = c_0\vec{u}_0 + c_1 e^{-t}\vec{u}_{-1} + c_2 e^{-2t}\vec{u}_{-2} + c_3 e^{-3t}\vec{u}_{-3}}$$

Not required:

$x = c_0 + c_1 e^{-t} + c_2 e^{-2t} - c_3 e^{-3t}$

$p = \quad - c_1 e^{-t} - 2c_2 e^{-2t} + 3c_3 e^{-3t}$

$y = c_0 + c_1 e^{-t} - c_2 e^{-2t} - c_3 e^{-3t}$

$q = \quad - c_1 e^{-t} + 2c_2 e^{-2t} + 3c_3 e^{-3t}$

6. (20 points) Consider the first order **non-homogeneous** system of differential equations

$$\vec{x}' = A\vec{x} + \vec{f} \quad \text{where} \quad \vec{x} = \begin{pmatrix} x \\ p \\ y \\ q \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \\ 1 & 3 & -1 & 0 \end{pmatrix} \quad \text{and} \quad \vec{f} = \begin{pmatrix} 3 \\ -t \\ 0 \\ t \end{pmatrix}.$$

The general solution of the corresponding homogeneous differential equation is

$\vec{x} = c_0\vec{u}_0 + c_1 e^{-t}\vec{u}_{-1} + c_2 e^{-2t}\vec{u}_{-2} + c_3 e^{3t}\vec{u}_3$ where

$$\vec{u}_0 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad u_{-1} = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} \quad \vec{u}_{-2} = \begin{pmatrix} 1 \\ -2 \\ -1 \\ 2 \end{pmatrix} \quad \vec{u}_3 = \begin{pmatrix} 1 \\ 3 \\ 1 \\ 3 \end{pmatrix}$$

Use undetermined coefficients to determine a **particular solution**.

HINTS: Look for a solution of the form $\vec{x}_p = \vec{a} + t\vec{b}$.

First solve for $\vec{b}$ and keep the arbitrary constant. Then solve for $\vec{a}$.

If you have time check your answer in the differential equation.

$$\text{Let } \vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}, \quad \vec{c} = \begin{pmatrix} 3 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \vec{d} = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

We substitute into the non-homogeneous equation:

$\vec{x}_p' = \vec{b} \quad = \quad A\vec{x}_p + \vec{f} = A\vec{a} + tA\vec{b} + \vec{c} + t\vec{d}$

We equate coefficients of t and 1:

$0 = A\vec{b} + \vec{d}$ and $\vec{b} = A\vec{a} + \vec{c}$

We solve $A\vec{b} = -\vec{d}$ for $\vec{b}$ and then $A\vec{a} = \vec{b} - \vec{c}$ for $\vec{a}$:

$$\left(\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 3 & -1 & 0 & -1 \end{array}\right) \Rightarrow \left(\begin{array}{cccc|c} 1 & 0 & -1 & -3 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 3 & -1 & 0 & -1 \end{array}\right) \Rightarrow \left(\begin{array}{cccc|c} 1 & 0 & -1 & -3 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 3 & 0 & 3 & 0 \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{cccc|c} 1 & 0 & -1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{array}{c} b_1 - b_3 = -1 \\ b_2 = 0 \\ b_4 = 0 \\ 0 = 0 \end{array} \Rightarrow \vec{b} = \begin{pmatrix} s-1 \\ 0 \\ s \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{cccc|c} 0 & 1 & 0 & 0 & s-4 \\ -1 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 & s \\ 1 & 3 & -1 & 0 & 0 \end{array}\right) \Rightarrow \left(\begin{array}{cccc|c} 1 & 0 & -1 & -3 & 0 \\ 0 & 1 & 0 & 0 & s-4 \\ 0 & 0 & 0 & 1 & s \\ 1 & 3 & -1 & 0 & 0 \end{array}\right) \Rightarrow \left(\begin{array}{cccc|c} 1 & 0 & -1 & -3 & 0 \\ 0 & 1 & 0 & 0 & s-4 \\ 0 & 0 & 0 & 1 & s \\ 0 & 3 & 0 & 3 & 0 \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 3s \\ 0 & 1 & 0 & 0 & s-4 \\ 0 & 0 & 0 & 1 & s \\ 0 & 0 & 0 & 0 & -6s+12 \end{array}\right) \Rightarrow \begin{array}{c} a_1 - a_3 = 3s \\ a_2 = s-4 \\ a_4 = s \\ 0 = -6s+12 \end{array} \Rightarrow s = 2$$

$$\vec{a} = \begin{pmatrix} r+6 \\ -2 \\ r \\ 2 \end{pmatrix} \quad \vec{b} = \begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \end{pmatrix} \Rightarrow \boxed{\vec{x}_p = \vec{a} + t\vec{b} = \begin{pmatrix} r+6 \\ -2 \\ r \\ 2 \end{pmatrix} + t\begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \end{pmatrix}}$$

Pick any r.

Check:

$$\vec{x}_p' = \begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \end{pmatrix}$$

$$A\vec{x}_p + \vec{f} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \\ 1 & 3 & -1 & 0 \end{pmatrix}\left(\begin{pmatrix} r+6 \\ -2 \\ r \\ 2 \end{pmatrix} + t\begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \end{pmatrix}\right) + \begin{pmatrix} 3 \\ -t \\ 0 \\ t \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \end{pmatrix}$$