1	/60
2	/40

MATH 308   Exam II   Fall 2000

Section 200   Solution   P. Yasskin

HAND COMPUTATIONS

1. (60 points) Solve the initial value problem

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = \cos(\omega t) \quad \text{with} \quad y(0) = \frac{3}{5} \quad \text{and} \quad y'(0) = \frac{1}{10}.$$

Then identify the steady state solution and any resonances. Be sure to follow each of the following steps:

a. (2 pts) Identify the homogeneous equation.

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = 0$$

b. (6 pts) Find the characteristic equation and the characteristic roots.

$$\lambda^2 + 2\lambda + 5 = 0 \quad \Rightarrow \quad \lambda = \frac{-2 \pm \sqrt{4 - 20}}{2} = -1 \pm 2i$$

c. (3 pts) Is the homogeneous equation overdamped, underdamped or critically damped? Why?

It is underdamped because the characteristic values are complex.

d. (6 pts) Find a fundamental set of solutions of the homogeneous equation and verify they are linearly independent by computing the Wronskian.

$$y_1 = e^{-t}\cos(2t) \qquad y_2 = e^{-t}\sin(2t)$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-t}\cos(2t) & e^{-t}\sin(2t) \\ -e^{-t}\cos(2t) - 2e^{-t}\sin(2t) & -e^{-t}\sin(2t) + 2e^{-t}\cos(2t) \end{vmatrix}$$

$$= e^{-t}\cos(2t)[-e^{-t}\sin(2t) + 2e^{-t}\cos(2t)] - e^{-t}\sin(2t)[-e^{-t}\cos(2t) - 2e^{-t}\sin(2t)]$$

$$= 2e^{-2t}\cos^2(2t) + 2e^{-2t}\sin^2(2t) = 2e^{-2t} \neq 0$$

Since $W \neq 0$ the solutions are linearly independent.

e. (2 pts) Write out the general solution to the homogeneous equation.

$$y_h = Ae^{-t}\cos(2t) + Be^{-t}\sin(2t)$$

f. (6 pts) To find the particular solution using the Method of Undetermined Coefficients, write out the general form of the particular solution. (Take the undetermined coefficients to be the coefficients of sin and cos **not** the amplitude and phase.) Then compute the first and second derivatives.

$$y_p = C\sin(\omega t) + D\cos(\omega t)$$
$$y_p' = \omega C\cos(\omega t) - \omega D\sin(\omega t)$$
$$y_p'' = -\omega^2 C\sin(\omega t) - \omega^2 D\cos(\omega t)$$

g. (8 pts) Using the Method of Undetermined Coefficients, find a particular solution to the inhomogeneous equation. (If you can't do this for general ω, at least do it for $\omega = 1$.)

Plug y_p into the inhomogeneous equation:
$$y_p'' + 2y_p' + 5y = \cos(\omega t)$$

$$[-\omega^2 C\sin(\omega t) - \omega^2 D\cos(\omega t)] + 2[\omega C\cos(\omega t) - \omega D\sin(\omega t)] + 5[C\sin(\omega t) + D\cos(\omega t)] = \cos(\omega t)$$

Equate coefficients of $\sin(\omega t)$ and $\cos(\omega t)$:

$\sin(\omega t)$:	$-\omega^2 C - 2\omega D + 5C = 0$	$\Rightarrow$	$(5-\omega^2)C - 2\omega D = 0$
$\cos(\omega t)$:	$-\omega^2 D + 2\omega C + 5D = 1$	$\Rightarrow$	$2\omega C + (5-\omega^2)D = 1$

To find C:
$$(5-\omega^2)^2 C - 2\omega(5-\omega^2)D = 0$$
$$(2\omega)^2 C + 2\omega(5-\omega^2)D = 2\omega$$

add:
$$\left[(2\omega)^2 + (5-\omega^2)^2\right]C = 2\omega$$
$$C = \frac{2\omega}{(2\omega)^2 + (5-\omega^2)^2}$$

To find D:
$$-2\omega(5-\omega^2)C + (2\omega)^2 D = 0$$
$$2\omega(5-\omega^2)C + (5-\omega^2)^2 D = (5-\omega^2)$$

add:
$$\left[(2\omega)^2 + (5-\omega^2)^2\right]D = (5-\omega^2)$$
$$D = \frac{(5-\omega^2)}{(2\omega)^2 + (5-\omega^2)^2}$$

$$y_p = \frac{2\omega}{(2\omega)^2 + (5-\omega^2)^2}\sin(\omega t) + \frac{(5-\omega^2)}{(2\omega)^2 + (5-\omega^2)^2}\cos(\omega t)$$

h. (2 pts) Write out the general solution to the inhomogeneous equation.

$$y = Ae^{-t}\cos(2t) + Be^{-t}\sin(2t) + \frac{2\omega}{(2\omega)^2 + (5-\omega^2)^2}\sin(\omega t) + \frac{(5-\omega^2)}{(2\omega)^2 + (5-\omega^2)^2}\cos(\omega t)$$

i. (6 pts) With $\omega = 1$, use the initial conditions to find the solution to the initial value problem with $y(0) = \frac{3}{5}$ and $y'(0) = \frac{1}{10}$.

$$y = Ae^{-t}\cos(2t) + Be^{-t}\sin(2t) + \frac{2}{(2)^2 + (5-1)^2}\sin(t) + \frac{(5-1)}{(2)^2 + (5-1)^2}\cos(t)$$

$$y = Ae^{-t}\cos(2t) + Be^{-t}\sin(2t) + \frac{1}{10}\sin(t) + \frac{1}{5}\cos(t)$$

$$y' = -Ae^{-t}\cos(2t) - 2Ae^{-t}\sin(2t) - Be^{-t}\sin(2t) + 2Be^{-t}\cos(2t) + \frac{1}{10}\cos(t) - \frac{1}{5}\sin(t)$$

$$y(0) = A + \frac{1}{5} = \frac{3}{5} \qquad \Rightarrow \qquad A = \frac{2}{5}$$

$$y'(0) = -A + 2B + \frac{1}{10} = \frac{1}{10} \qquad \Rightarrow \qquad B = \frac{1}{2}A = \frac{1}{5}$$

$$y = \frac{2}{5}e^{-t}\cos(2t) + \frac{1}{5}e^{-t}\sin(2t) + \frac{1}{10}\sin(t) + \frac{1}{5}\cos(t)$$

j. (4 pts) With ω once again general, for large times, what is the steady state solution? Why?

At large times the exponentials all approach zero and the steady state solution is

$$y = \frac{2\omega}{(2\omega)^2 + (5 - \omega^2)^2}\sin(\omega t) + \frac{(5 - \omega^2)}{(2\omega)^2 + (5 - \omega^2)^2}\cos(\omega t)$$

k. (6 pts) Find the amplitude of the steady state solution.

With $C = \dfrac{2\omega}{(2\omega)^2 + (5 - \omega^2)^2}$ and $D = \dfrac{(5 - \omega^2)}{(2\omega)^2 + (5 - \omega^2)^2}$:

$$A = \sqrt{D^2 + C^2} = \sqrt{\left(\frac{(5 - \omega^2)}{(2\omega)^2 + (5 - \omega^2)^2}\right)^2 + \left(\frac{2\omega}{(2\omega)^2 + (5 - \omega^2)^2}\right)^2} = \frac{\sqrt{(2\omega)^2 + (5 - \omega^2)^2}}{(2\omega)^2 + (5 - \omega^2)^2}$$

$$= \frac{1}{\sqrt{(2\omega)^2 + (5 - \omega^2)^2}}$$

l. (6 pts) Identify the resonance (angular) frequency ω_R (if any) by maximizing the steady state amplitude.

$$A' = -\frac{1}{2}\frac{2(2\omega)2 + 2(5 - \omega^2)(-2\omega)}{\left[(2\omega)^2 + (5 - \omega^2)^2\right]^{3/2}} = 0 \qquad \Rightarrow \qquad 8\omega - 20\omega + 4\omega^3 = 0$$

$$\Rightarrow \qquad -3\omega + \omega^3 = 0 \qquad \Rightarrow \qquad \omega = 0, \sqrt{3}, -\sqrt{3} \qquad \Rightarrow \qquad \omega_R = \sqrt{3}$$

m. (3 pts) What is the steady state amplitude at resonance?

$$A_R = \frac{1}{\sqrt{(2\omega_R)^2 + (5 - \omega_R^2)^2}} = \frac{1}{\sqrt{\left(2\sqrt{3}\right)^2 + \left(5 - \sqrt{3}^2\right)^2}} = \frac{1}{\sqrt{12 + 4}} = \frac{1}{4}$$

MAPLE COMPUTATIONS

2. (40 points) Consider the initial value problem

$$\frac{d^2y}{dt^2} + b\frac{dy}{dt} + 5y = \cos(\omega t)$$

for general b and ω.

a. (3 pts) For what values of b is the system overdamped, underdamped or critically damped? Hereafter, assume the system is underdamped and $b > 0$.

b. (2 pts) Find the general solution in terms of b and ω.

c. (4 pts) Find the specific solution for $b = 2$ and $\omega = 3$ satisfying the initial conditions $y(0) = 2$ and $y'(0) = 1$. Plot this solution.

d. (8 pts) Find the steady state solution for general b and ω and then find its amplitude and phase shift.

e. (5 pts) Plot the steady state amplitude as a function of ω for $b = 1,2,3,4$.

f. (5 pts) Identify the resonance (angular) frequency ω_R (if any) as a function of b by maximizing the steady state amplitude.

g. (3 pts) For what values of b is there a resonance? For what values of b is there no resonance? Does this agree with your plot?

h. (5 pts) What is the steady state amplitude at resonance as a function of b?

i. (5 pts) Plot the resonant steady state amplitude as a function of b.

To Turn in Your Maple Computations:
1. Save your Maple file as lastname_exam2.mws
2. Print your file as follows:
   a. Click on **FILE**, **PRINT** and **Printer Command**.
   b. Make the command read: **lpr -J** ”**Yasskin Maple Exam 2**”
   c. Call Dr. Yasskin over to check your printing.
   d. Click on **PRINT**.
3. Mail your file as follows:
   a. Start the mail program: **pine**
   b. Compose a letter by typing   **C**.
   c. In the header region, enter:
      To   **yasskin**
      Attachment **lastname_exam2.mws** (or the *exact* name of your Maple file)
      Subject **Last Name** **Exam2**
   d. Call Dr. Yasskin over to check your email.
   e. Mail the letter by typing   **^X** and **Y**.