Name________________________ ID______________

1	/15
2	/40
3	/15
4	/30

MATH 308 Final Fall 2000

Section 200 Solutions P. Yasskin

HAND COMPUTATIONS

1. (15 points) Find the inverse Laplace transform of the function

$$F(s) = \frac{(s-2)e^{-3s}}{(s-2)^2+25} + \frac{4}{s-1} + \frac{3}{(s-1)^2}.$$

$$F_1 = \frac{s}{s^2+25} \Leftrightarrow f_1 = \cos 5t \qquad F_4 = \frac{1}{s-1} \Leftrightarrow f_4 = e^t$$

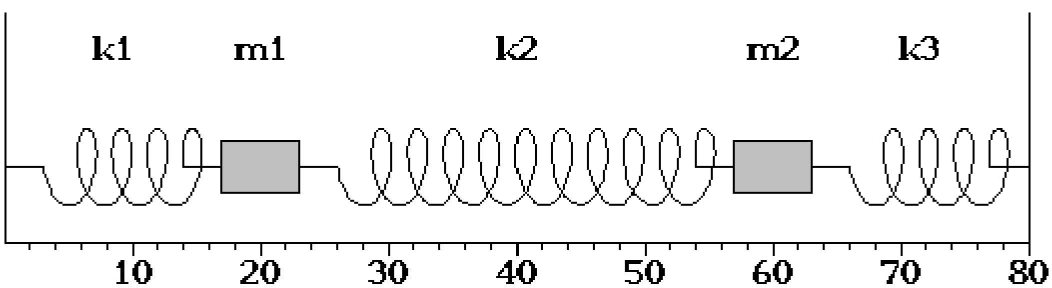
$$F_2 = \frac{s-2}{(s-2)^2+25} \Leftrightarrow f_2 = e^{2t}\cos 5t \qquad F_5 = \frac{1}{s^2} \Leftrightarrow f_5 = t$$

$$F_3 = \frac{(s-2)e^{-3s}}{(s-2)^2+25} \Leftrightarrow f_3 = e^{2(t-3)}\cos 5(t-3)\Theta(t-3) \qquad F_6 = \frac{1}{(s-1)^2} \Leftrightarrow f_6 = e^t t$$

$$F = F_3 + 4F_4 + 3F_6 \quad \Leftrightarrow \quad f = f_3 + 4f_4 + 3f_6 = e^{2(t-3)}\cos 5(t-3)\Theta(t-3) + 4e^t + 3e^t t$$

If you want me to verify your answer to any part of the following problem, call me over. If you are wrong, I will give you the correct answer and you will lose part or all of the indicated points but you will be able to proceed to subsequent parts.

2. (40 points) Consider the 2 mass, 3 spring system shown below.



The masses and spring constants are

$$m_1 = m_2 = 1 \text{ gm}, \quad k_1 = k_3 = 4\ \frac{\text{gm}}{\text{sec}^2}, \quad k_2 = 6\ \frac{\text{gm}}{\text{sec}^2}$$

The walls are at $x = 0$ and $x = 80$ cm. The rest positions of the masses are at $x_{1r} = 20$ cm and $x_{2r} = 60$ cm. Let $x_1(t)$ and $x_2(t)$ be the positions of the masses at time t in sec. The initial conditions are

$$x_1(0) = 20 \text{ cm}, \qquad x_2(0) = 62 \text{ cm}, \qquad x_1'(0) = 16\ \frac{\text{cm}}{\text{sec}}, \qquad x_2'(0) = 0\ \frac{\text{cm}}{\text{sec}}$$

a. (3 pts) Write out the second order differential equations satisfied by $x_1(t)$ and $x_2(t)$.

$$m_1\frac{d^2x_1}{dt^2} = -k_1(x_1-20) - k_2(x_1-20) + k_2(x_2-60)$$
$$m_2\frac{d^2x_2}{dt^2} = -k_2(x_2-60) + k_2(x_1-20) - k_3(x_2-60)$$

$$\frac{d^2x_1}{dt^2} = -10(x_1-20) + 6(x_2-60)$$
$$\frac{d^2x_2}{dt^2} = 6(x_1-20) - 10(x_2-60)$$

b. (4 pts) To simplify the equations make the change of variables from $x_1(t)$ and $x_2(t)$ to

$$y_1(t) = x_1(t) - 20 \quad \text{and} \quad y_2(t) = x_2(t) - 60$$

which measure the distance of each mass from its rest position. Write out the second order differential equations and the initial conditions satisfied by $y_1(t)$ and $y_2(t)$.

$$\frac{d^2y_1}{dt^2} = -10y_1 + 6y_2 \qquad y_1(0) = 0 \qquad y_1'(0) = 16$$
$$\frac{d^2y_2}{dt^2} = 6y_1 - 10y_2 \qquad y_2(0) = 2 \qquad y_2'(0) = 0$$

c. (10 pts) Find the Laplace transforms $Y_1(s)$ and $Y_2(s)$ of the solutions $y_1(t)$ and $y_2(t)$. DO NOT simplify or invert the transforms.

$$s^2Y_1 - sy_1(0) - y_1'(0) = -10Y_1 + 6Y_2 \qquad s^2Y_2 - sy_2(0) - y_2'(0) = 6Y_1 - 10Y_2$$
$$s^2Y_1 - 16 = -10Y_1 + 6Y_2 \qquad s^2Y_2 - 2s = 6Y_1 - 10Y_2$$
$$(s^2+10)Y_1 - 6Y_2 = 16 \qquad -6Y_1 + (s^2+10)Y_2 = 2s$$

Cross multiply and add:

$$\left[(s^2+10)^2 - 36\right]Y_1 = 16(s^2+10) + 12s \qquad \left[(s^2+10)^2 - 36\right]Y_2 = 2s(s^2+10) + 96$$
$$Y_1 = \frac{16(s^2+10)+12s}{(s^2+10)^2-36} \qquad Y_2 = \frac{2s(s^2+10)+96}{(s^2+10)^2-36}$$
$$Y_1 = \frac{16s^2+12s+160}{s^4+20s^2+64} \qquad Y_2 = \frac{2s^3+20s+96}{s^4+20s^2+64}$$

d. (3 pts) Now let $v_1(t) = \frac{dy_1}{dt}$ and $v_2(t) = \frac{dy_2}{dt}$. Write out a system of first order differential equations for $y_1(t)$, $y_2(t)$, $v_1(t)$ and $v_2(t)$. (Please list the equations and variables in this order.)

$$\frac{dy_1}{dt} = v_1$$
$$\frac{dy_2}{dt} = v_2$$
$$\frac{dv_1}{dt} = -10y_1 + 6y_2$$
$$\frac{dv_2}{dt} = 6y_1 - 10y_2$$

e. (10 pts) Identify the coefficient matrix of the first order system.
(Please list the variables in the order y_1, y_2, v_1 and v_2.
HINT: How many rows and columns are there in the coefficient matrix?)
Find its eigenvalues. Find the eigenvector for **one** eigenvalue.
(DO NOT find the other three eigenvectors.)

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -10 & 6 & 0 & 0 \\ 6 & -10 & 0 & 0 \end{pmatrix} \qquad A - \lambda\mathbf{1} = \begin{pmatrix} -\lambda & 0 & 1 & 0 \\ 0 & -\lambda & 0 & 1 \\ -10 & 6 & -\lambda & 0 \\ 6 & -10 & 0 & -\lambda \end{pmatrix}$$

$$\det(A - \lambda\mathbf{1}) = -\lambda \begin{vmatrix} -\lambda & 0 & 1 \\ 6 & -\lambda & 0 \\ -10 & 0 & -\lambda \end{vmatrix} + 1 \begin{vmatrix} 0 & -\lambda & 1 \\ -10 & 6 & 0 \\ 6 & -10 & -\lambda \end{vmatrix}$$

$$= -\lambda(-\lambda^3 - 10\lambda) + 1(100 + 10\lambda^2 - 36) = \lambda^4 + 20\lambda^2 + 64 = 0$$

$$\lambda^2 = \frac{-20 \pm \sqrt{400 - 256}}{2} = -10 \pm 6 = -4, -16$$

$$\lambda = \pm\sqrt{-4}, \quad \pm\sqrt{-16} = 2i, -2i, 4i, -4i$$

Look at $\lambda = 2i$:

$$\begin{pmatrix} -2i & 0 & 1 & 0 \\ 0 & -2i & 0 & 1 \\ -10 & 6 & -2i & 0 \\ 6 & -10 & 0 & -2i \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{aligned} -2iu_1 + u_3 &= 0 \\ -2iu_2 + u_4 &= 0 \\ -10u_1 + 6u_2 - 2iu_3 &= 0 \\ 6u_1 - 10u_2 - 2iu_4 &= 0 \end{aligned}$$

$$\vec{u}_{2i} = u_1 \begin{pmatrix} 1 \\ 1 \\ 2i \\ 2i \end{pmatrix} \qquad \vec{u}_{-2i} = u_1 \begin{pmatrix} 1 \\ 1 \\ -2i \\ -2i \end{pmatrix} \qquad \vec{u}_{4i} = u_1 \begin{pmatrix} 1 \\ -1 \\ 4i \\ -4i \end{pmatrix} \qquad \vec{u}_{-4i} = u_1 \begin{pmatrix} 1 \\ -1 \\ -4i \\ 4i \end{pmatrix}$$

f. (10 pts) Now suppose an external force $F = 35\cos(3t)$ is applied to mass 1 only. Write out the modified second order differential equations for $y_1(t)$ and $y_2(t)$. Using the method of undetermined coefficients, find a particular solution for $y_1(t)$ and $y_2(t)$. (In other words, guess the form of $y_1(t)$ and $y_2(t)$ based on the form of the external force, plug into the equations and solve for the coefficients.)

$$\frac{d^2y_1}{dt^2} = -10y_1 + 6y_2 + 35\cos 3t$$
$$\frac{d^2y_2}{dt^2} = 6y_1 - 10y_2$$

$$y_{1p} = A\sin 3t + B\cos 3t \qquad y_{1p}'' = -9A\sin 3t - 9B\cos 3t$$
$$y_{2p} = C\sin 3t + D\cos 3t \qquad y_{2p}'' = -9C\sin 3t - 9D\cos 3t$$

$$-9A\sin 3t - 9B\cos 3t = -10(A\sin 3t + B\cos 3t) + 6(C\sin 3t + D\cos 3t) + 35\cos 3t$$
$$-9C\sin 3t - 9D\cos 3t = 6(A\sin 3t + B\cos 3t) - 10(C\sin 3t + D\cos 3t)$$

Coefficient of $\sin 3t$:

$$\begin{array}{lcl} -9A = -10A + 6C & \Rightarrow & A = 6C \\ -9C = 6A - 10C & \Rightarrow & C = 6A \end{array} \Rightarrow \begin{array}{l} A = 0 \\ C = 0 \end{array}$$

Coefficient of $\cos 3t$:

$$\begin{array}{lcl} -9B = -10B + 6D + 35 & \Rightarrow & B - 6D = 35 \\ -9D = 6B - 10D & \Rightarrow & D = 6B \end{array} \Rightarrow \begin{array}{l} B = -1 \\ D = -6 \end{array}$$

Particular solution:

$$y_{1p} = -\cos 3t \qquad y_{2p} = -6\cos 3t$$

3. (15 points) There are two tanks of water. Tank A initially holds 20 gal of sugar water with a concentration of 0.3 lb per gal. Tank B initially holds 30 gal of pure water. Pure water flows into tank A at the rate of 3 gal/hr and sugar water from tank A flows into tank B at the rate of 4 gal/hr. Sugar water from tank B is pumped back into tank A at the rate of 1 gal/hr and sugar water is drained from tank B at the rate of 3 gal /hr.

Set up the differential equations and initial conditions for the amount of sugar in each tank. Be sure to say what your variables mean. DO NOT solve it. Draw a rough qualitative graph of the amount of sugar in each tank as a function of time.

Let $A(t)$ be the lbs of sugar in tank A. Let $B(t)$ be the lbs of sugar in tank B.

Differential Equations:

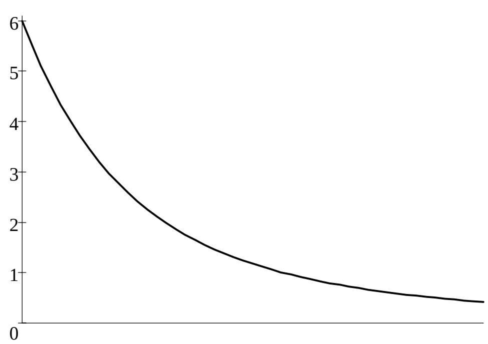
$$\frac{dA}{dt} = \underbrace{\frac{1\text{gal}}{\text{hr}} \bullet \frac{B\text{ lb}}{30\text{gal}}}_{\text{in}} - \underbrace{\frac{4\text{gal}}{\text{hr}} \bullet \frac{A\text{ lb}}{20\text{gal}}}_{\text{out}} = -\frac{1}{5}A + \frac{1}{30}B$$

$$\frac{dB}{dt} = \underbrace{\frac{4\text{gal}}{\text{hr}} \bullet \frac{A\text{ lb}}{20\text{gal}}}_{\text{in}} - \underbrace{\frac{1\text{gal}}{\text{hr}} \bullet \frac{B\text{ lb}}{30\text{gal}}}_{\text{out}} - \underbrace{\frac{3\text{gal}}{\text{hr}} \bullet \frac{B\text{ lb}}{30\text{gal}}}_{\text{out}} = \frac{1}{5}A - \frac{2}{15}B$$

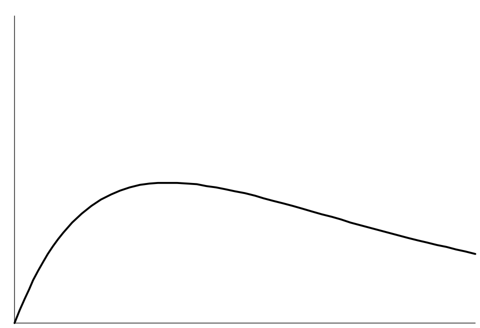
Initial Conditions:

$$A(0) = 20\text{gal} \bullet 0.3\frac{\text{lb}}{\text{gal}} = 6\text{lb}$$

$$B(0) = 0$$



$A(t)$



$B(t)$

MAPLE COMPUTATIONS

4. (30 points) Again consider the spring-mass system of problem 2. In part 2(b), you should have obtained the differential equations

$$\frac{d^2y_1}{dt^2} = -10y_1 + 6y_2$$
$$\frac{d^2y_2}{dt^2} = 6y_1 - 10y_2$$

and the initial conditions

$$y_1(0) = 0, \qquad y_2(0) = 2 \text{ cm}, \qquad y_1'(0) = 16\,\frac{\text{cm}}{\text{sec}}, \qquad y_2'(0) = 0.$$

a. In part 2(c), you should have found the Laplace transforms of the solutions are

$$Y_1(s) = \frac{16s^2 + 12s + 160}{s^4 + 20s^2 + 64} \quad \text{and} \quad Y_2(s) = \frac{2s^3 + 20s + 96}{s^4 + 20s^2 + 64}$$

- **i.** (3 pts) Use the inverse Laplace transform to find the solutions $y_1(t)$ and $y_2(t)$.
- **ii.** (3 pts) Plot $y_1(t)$ and $y_2(t)$ for $0 \le t \le 4\pi$.
- **iii.** (2 pts) What is the period of oscillation? Answer in a sentence.

b. In part 2(e), you should have found the eigenvalues $\lambda = 2i, -2i, 4i, -4i$.

- **i.** (2 pts) How are these reflected in the solutions found in part 4(a)? Answer in a sentence.

c. In part 2(f), a force was applied to mass 1 and you should have obtained the differential equations

$$\frac{d^2y_1}{dt^2} = -10y_1 + 6y_2 + 35\cos(3t)$$
$$\frac{d^2y_2}{dt^2} = 6y_1 - 10y_2$$

- **i.** (3 pts) Using the simplest way possible in Maple, find the solution satisfying the initial conditions
$$y_1(0) = 0, \qquad y_2(0) = 2 \text{ cm}, \qquad y_1'(0) = 16\,\frac{\text{cm}}{\text{sec}}, \qquad y_2'(0) = 0.$$
- **ii.** (3 pts) Plot $y_1(t)$ and $y_2(t)$ for $0 \le t \le 4\pi$.
- **iii.** (2 pts) What is the period of oscillation? Answer in a sentence.
- **iv.** (2 pts) From the form of the solution, not the plot, why is the period not π? Answer in a sentence.

d. Now suppose the force on mass 1 is changed to $F = 24\cos(2t)$.

- **i.** (3 pts) Using the simplest way possible in Maple, find the solution satisfying the initial conditions
$$y_1(0) = 0, \qquad y_2(0) = 2 \text{ cm}, \qquad y_1'(0) = 16\,\frac{\text{cm}}{\text{sec}}, \qquad y_2'(0) = 0.$$
- **ii.** (3 pts) Plot $y_1(t)$ and $y_2(t)$ for $0 \le t \le 4\pi$.
- **iii.** (4 pts) What is happening to the solution? Why is the solution not periodic? Why is this solution unphysical? What would need to be changed in the differential equations to make them more physical. Answer in sentences.

To Turn in Your Maple Computations:

1. Save your Maple file as lastname_final.mws
2. Print your file as follows:
   a. Click on **FILE**, **PRINT** and **Printer Command**.
   b. Make the command read: **lpr** -**J** "**Yasskin Maple Final**"
   c. Call Dr. Yasskin over to check your printing.
   d. Click on **PRINT**.
3. Mail your file as follows:
   a. Start the mail program: **pine**
   b. Compose a letter by typing **C**.
   c. In the header region, enter:

      To **yasskin**
      Attachment **lastname_final**.**mws** (or the *exact* name of your Maple file)
      Subject **Last Name** **Final**

   d. Call Dr. Yasskin over to check your email.
   e. Mail the letter by typing ^**X** and **Y**.