MATH 311 Exam 1 Fall 2000
Section 502 Solutions P. Yasskin

1. (10 points) Find the inverse of $A = \begin{pmatrix} -1 & 0 & 3 \\ 0 & 1 & 4 \\ 1 & 1 & 0 \end{pmatrix}$.

$$\left(\begin{array}{ccc|ccc} -1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \begin{matrix} R3 \\ \\ R1 \end{matrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ -1 & 0 & 3 & 1 & 0 & 0 \end{array}\right) \begin{matrix} \\ \\ R3+R1 \end{matrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 1 & 3 & 1 & 0 & 1 \end{array}\right) \begin{matrix} R1-R2 \\ \\ R3-R2 \end{matrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -4 & 0 & -1 & 1 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & -1 & 1 \end{array}\right) \begin{matrix} \\ \\ -R3 \end{matrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -4 & 0 & -1 & 1 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & -1 \end{array}\right) \begin{matrix} R1+4R3 \\ R2-4R3 \\ \end{matrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -4 & 3 & -3 \\ 0 & 1 & 0 & 4 & -3 & 4 \\ 0 & 0 & 1 & -1 & 1 & -1 \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} -4 & 3 & -3 \\ 4 & -3 & 4 \\ -1 & 1 & -1 \end{pmatrix}$$

Use it to solve $XA = \begin{pmatrix} 0 & 0 & 2 \\ 2 & 1 & 0 \\ 0 & 2 & 0 \end{pmatrix}$.

$$XA = B \qquad \Rightarrow \qquad X = BA^{-1} = \begin{pmatrix} 0 & 0 & 2 \\ 2 & 1 & 0 \\ 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} -4 & 3 & -3 \\ 4 & -3 & 4 \\ -1 & 1 & -1 \end{pmatrix} = \begin{pmatrix} -2 & 2 & -2 \\ -4 & 3 & -2 \\ 8 & -6 & 8 \end{pmatrix}$$

2. (10 points) Consider the polynomials

$$p_1(x) = 1 - x^2$$
$$p_2(x) = 2 - x - x^2$$
$$p_3(x) = 1 - x$$

and the vector space

$$W = \text{Span}(p_1, p_2, p_3).$$

Find a subset of $\{p_1, p_2, p_3\}$ which is a basis for W. Prove it spans W and is linearly independent.

$$ap_1 + bp_2 + cp_3 = 0 \qquad \Rightarrow \qquad a(1-x^2) + b(2-x-x^2) + c(1-x) = 0$$

$$\Rightarrow \qquad -a-b = 0 \qquad -b-c = 0 \qquad a+2b+c = 0$$

$$\left(\begin{array}{ccc|c} -1 & -1 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 1 & 2 & 1 & 0 \end{array}\right) \begin{array}{l} -R1 \\ -R2 \\ R3+R1 \end{array} \quad \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right) \begin{array}{l} R1-R2 \\ \\ R3-R2 \end{array} \quad \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{l} R1-R2 \\ \\ R3-R2 \end{array} \quad \begin{array}{l} a = t \\ b = -t \\ c = t \end{array}$$

Let $t = 1$. Then

$$p_1 - p_2 + p_3 = 0 \qquad \text{or} \qquad p_3 = -p_1 + p_2$$

Claim the basis is $\{p_1, p_2\}$. They span because

$$ap_1 + bp_2 + cp_3 = ap_1 + bp_2 + c(-p_1 + p_2) = (a-c)p_1 + (b+c)p_2$$

They are linearly independent because

$$ap_1 + bp_2 = 0 \qquad \Rightarrow \qquad a(1-x^2) + b(2-x-x^2) = 0$$

$$\Rightarrow \qquad -a-b = 0 \qquad -b = 0 \qquad a+2b = 0$$

$$\Rightarrow \qquad b = 0 \quad \& \quad a = 0$$

3. Consider the vector space P_3, the set of polynomials of degree 3 or less?

- (5 points) Scantron #1 Which of the following is NOT a subspace of P_3?
  - **a**. $A = \{ \ p \in P_3 \ | \ p(0) = 0 \ \}$
  - **b**. $B = \{ \ p \in P_3 \ | \ p(1) = 0 \ \}$
  - **c**. $C = \{ \ p \in P_3 \ | \ p(0) = p(1) \ \}$
  - **d**. $D = \{ \ p \in P_3 \ | \ p(0) + p(1) = 0 \ \}$
  - **e**. $E = \{ \ p \in P_3 \ | \ p(0) = 1 \ \}$ Correct

E is not a subspace because if $p, q \in E$ then $p(0) = 1$ and $q(0) = 1$. So $(p+q)(0) = 2$ and $p + q \notin E$.

4. Consider the vector space $\mathbf{R}^+$ of all positive real numbers with the operations of

Vector Addition: $x \oplus y = xy$ (real number addition)

Scalar Multiplication: $\alpha \circ x = x^\alpha$ (real number exponentiation)

- (5 points) Scantron #2 Translate the vector identity

$$0 \circ x = \vec{0}$$

into ordinary arithmetic.

a. $1^x = 1$
b. $x^0 = 1$ Correct
c. $0^x = 0$
d. $x^1 = x$
e. $0^x = 1$

$0 \circ x = x^0$ and $\vec{0} = 1.$ So $x^0 = 1.$

5. Consider the linear map $L : \mathbf{R}^3 \to \mathbf{R}^4$ given by $L(\vec{x}) = A\vec{x}$ where $A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 2 \\ 2 & 0 & 4 \\ 3 & -1 & 4 \end{pmatrix}$.

- (10 points) Solve $L(\vec{x}) = \begin{pmatrix} 2 \\ -1 \\ 2 \\ 4 \end{pmatrix}$.

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 2 \\ 0 & 1 & 2 & -1 \\ 2 & 0 & 4 & 2 \\ 3 & -1 & 4 & 4 \end{array}\right) \begin{array}{l} \\ \\ R3 - 2R1 \\ R4 - 3R1 \end{array} \quad \left(\begin{array}{ccc|c} 1 & -1 & 0 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 2 & 4 & -2 \\ 0 & 2 & 4 & -2 \end{array}\right) \begin{array}{l} R1 + R2 \\ \\ R3 - 2R2 \\ R4 - 2R2 \end{array} \quad \left(\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\Rightarrow \quad \begin{array}{l} x + 2z = 1 \\ y + 2z = -1 \end{array} \quad \Rightarrow \quad \begin{array}{l} x = 1 - 2t \\ y = -1 - 2t \\ z = t \end{array}$$

- (5 points) Scantron #3 Describe the solution set:

a. No Solutions
b. Unique Solution (Point in $\mathbf{R}^3$)
c. ∞-Many Solutions (Line in $\mathbf{R}^3$) Correct There is one parameter.
d. ∞-Many Solutions (Plane in $\mathbf{R}^3$)
e. ∞-Many Solutions (All of $\mathbf{R}^3$)

- (5 points) Scantron #4 Is L a one-to-one function?

a. Yes
b. No Correct There is more than one solution to $L(\vec{x}) = \vec{b}$.

6. Again consider the linear map $L : \mathbf{R}^3 \to \mathbf{R}^4$ given by $L(\vec{x}) = A\vec{x}$ where $A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 2 \\ 2 & 0 & 4 \\ 3 & -1 & 4 \end{pmatrix}$.

- (10 points) Solve $L(\vec{x}) = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$.

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & 2 & 1 \\ 2 & 0 & 4 & 1 \\ 3 & -1 & 4 & 1 \end{array}\right) \begin{array}{l} \\ \\ R3 - 2R1 \\ R4 - 3R1 \end{array} \quad \left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 2 & 4 & -1 \\ 0 & 2 & 4 & -2 \end{array}\right) \begin{array}{l} R1 + R2 \\ \\ R3 - 2R2 \\ R4 - 2R2 \end{array} \quad \left(\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & -4 \end{array}\right)$$

$\Rightarrow$ No Solution

- (5 points) Scantron #5 Describe the solution set:
  - **a.** No Solutions Correct
  - **b.** Unique Solution (Point in $\mathbf{R}^3$)
  - **c.** ∞-Many Solutions (Line in $\mathbf{R}^3$)
  - **d.** ∞-Many Solutions (Plane in $\mathbf{R}^3$)
  - **e.** ∞-Many Solutions (All of $\mathbf{R}^3$)

- (5 points) Scantron #6 Is L an onto function?
  - **a.** Yes
  - **b.** No Correct $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ is not in the image.

7. Again consider the linear map $L : \mathbf{R}^3 \to \mathbf{R}^4$ given by $L(\vec{x}) = A\vec{x}$ where $A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 2 \\ 2 & 0 & 4 \\ 3 & -1 & 4 \end{pmatrix}$.

- (5 points) Find $Ker(L)$, the kernel (or null space) of L.

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 2 & 0 & 4 & 0 \\ 3 & -1 & 4 & 0 \end{array}\right) \begin{array}{l} \\ \\ R3 - 2R1 \\ R4 - 3R1 \end{array} \quad \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 2 & 4 & 0 \end{array}\right) \begin{array}{l} R1 + R2 \\ \\ R3 - 2R2 \\ R4 - 2R2 \end{array} \quad \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\Rightarrow \quad \begin{array}{l} x + 2z = 0 \\ y + 2z = 0 \end{array} \quad \Rightarrow \quad \begin{array}{l} x = -2t \\ y = -2t \\ z = t \end{array} \quad \Rightarrow \quad Ker(L) = \left\{ \begin{pmatrix} -2t \\ -2t \\ t \end{pmatrix} \right\}$$

- (5 points) Give a basis for $Ker(L)$. (No proof) $\quad \left\{ \begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix} \right\}$

- (5 points) What is the dimension of $Ker(L)$? (No proof) $\quad \dim Ker(L) = 1$

8. Again consider the linear map $L : \mathbf{R}^3 \to \mathbf{R}^4$ given by $L(\vec{x}) = A\vec{x}$ where $A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 2 \\ 2 & 0 & 4 \\ 3 & -1 & 4 \end{pmatrix}$.

- (5 points) Find $Im(L)$, the image (or range) of L.

$$L(\vec{x}) = A\vec{x} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 2 \\ 2 & 0 & 4 \\ 3 & -1 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x-y \\ y+2z \\ 2x+4z \\ 3x-y+4z \end{pmatrix} = x\begin{pmatrix} 1 \\ 0 \\ 2 \\ 3 \end{pmatrix} + y\begin{pmatrix} -1 \\ 1 \\ 0 \\ -1 \end{pmatrix} + z\begin{pmatrix} 0 \\ 2 \\ 4 \\ 4 \end{pmatrix}$$

$$Im(L) = \{L(\vec{x})\} = Span\left(\begin{pmatrix} 1 \\ 0 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 4 \\ 4 \end{pmatrix}\right)$$

- (5 points) Give a basis for $Im(L)$. (No proof)

The 3 vectors span. Are they independent?

$$x\begin{pmatrix} 1 \\ 0 \\ 2 \\ 3 \end{pmatrix} + y\begin{pmatrix} -1 \\ 1 \\ 0 \\ -1 \end{pmatrix} + z\begin{pmatrix} 0 \\ 2 \\ 4 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 2 & 0 & 4 & 0 \\ 3 & -1 & 4 & 0 \end{array}\right) \quad \Rightarrow \quad \begin{array}{l} x = -2t \\ y = -2t \\ z = t \end{array}$$

as in Problem 7. So they are not independent. Throw out the third vector and the first two are independent.

Basis is $\left\{\begin{pmatrix} 1 \\ 0 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \\ -1 \end{pmatrix}\right\}$

- (5 points) What is the dimension of $Im(L)$? (No proof) $\dim Im(L) = 2$