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1-5	/25	7	/11
6	/45	8	/24

MATH 311 Final Exam Fall 2000
Section 502 Solutions P. Yasskin

1. (5 points) Compute $\oint (4x-3y)\,dx + (3x-2y)\,dy$ counterclockwise around the edge of the triangle with vertices $(0,0)$, $(0,3)$ and $(2,0)$.
(HINT: Use Green's Theorem.)

a. 3
b. 6
c. 9
d. 12
e. 18 correctchoice

$P = 4x-3y \qquad Q = 3x-2y$

$$\oint P\,dx + Q\,dy = \iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy = \iint (3-(-3))\,dx\,dy = 6 \bullet Area = 6\left(\frac{1}{2} \bullet 2 \bullet 3\right) = 18$$

2. (5 points) Let $M(2,2)$ be the vector space of 2×2 matrices. Consider the linear map

$L : M(2,2) \to M(2,2)$ given by $L(X) = X + X^{\mathsf{T}}$

where X^{T} is the transpose of X. Which of the following is FALSE or are they all TRUE?

a. Domain$(L) = M(2,2)$
b. Codomain$(L) = M(2,2)$
c. Kernel$(L) = \{$antisymmetric 2×2 matrices$\}$
d. Image$(L) = \{$symmetric 2×2 matrices$\}$
e. All of the above are TRUE correctchoice

The domain and range are before and after the arrow in $L : M(2,2) \to M(2,2)$.
The kernel is the set of solutions of $L(X) = X + X^{\mathsf{T}} = 0$. But then $X^{\mathsf{T}} = -X$ and X is antisymmetric.
The image is the set of matrices of the form $X + X^{\mathsf{T}}$ which is symmetric. But is every symmetric matrix of this form? Yes because if S is symmetric, then $S^{\mathsf{T}} = S$ and $L\left(\frac{1}{2}S\right) = \frac{1}{2}S + \frac{1}{2}S^{\mathsf{T}} = S$.
So (a-d) are true.

3. (5 points) If a jet flies around the world from East to West, directly above the equator, in what direction does the unit binormal $\hat{B}$ point?
(HINT: $\hat{B} = \hat{T} \times \hat{N}$)

a. North
b. South correctchoice
c. East
d. Up (away from the center of the earth)
e. Down (toward the center of the earth)

$\hat{T}$ points West $\quad \hat{N}$ points Down $\quad$ So $\hat{B}$ points South by the right hand rule.

4. (5 points) Let P_2 be the vector space of polynomials of degree at most 2. Consider the inner product

$$\langle p,q\rangle = \int_0^1 3x\,p(x)\,q(x)\,dx$$

Find the angle between the polynomials $r(x) = 1 - x^2$ and $s(x) = x$.

- **a.** $\cos^{-1}\left(\dfrac{4\sqrt{2}}{5\sqrt{3}}\right)$ correctchoice
- **b.** $\cos^{-1}\left(\frac{16}{15}\right)$
- **c.** $\cos^{-1}\left(\dfrac{5\sqrt{3}}{4\sqrt{2}}\right)$
- **d.** $\cos^{-1}\left(\frac{15}{16}\right)$
- **e.** $\cos^{-1}\left(\dfrac{4\sqrt{3}}{5\sqrt{2}}\right)$

$$|r|^2 = \int_0^1 3x\,r(x)^2\,dx = \int_0^1 3x(1-x^2)^2\,dx = -\frac{(1-x^2)^3}{2}\bigg|_0^1 = 0 - \left(-\frac{1}{2}\right) = \frac{1}{2}$$

$$|s|^2 = \int_0^1 3x\,s(x)^2\,dx = \int_0^1 3x^3\,dx = \frac{3x^4}{4}\bigg|_0^1 = \frac{3}{4}$$

$$\langle r,s\rangle = \int_0^1 3x\,r(x)\,s(x)\,dx = \int_0^1 3x(1-x^2)(x)\,dx = \int_0^1 (3x^2 - 3x^4)\,dx = \left[x^3 - \frac{3x^5}{5}\right]_0^1 = 1 - \frac{3}{5} = \frac{2}{5}$$

$$\cos\theta = \frac{\langle r,s\rangle}{|r||s|} = \frac{\frac{2}{5}}{\sqrt{\frac{1}{2}}\sqrt{\frac{3}{4}}} = \frac{2}{5}\sqrt{\frac{8}{3}} = \frac{4\sqrt{2}}{5\sqrt{3}}$$

5. (5 points) Compute $\iint\limits_{\partial C} \vec{F}\bullet d\vec{S}$ for $\vec{F} = (x^3, y^3, x^2z + y^2z)$ over the complete surface of the cylinder $x^2 + y^2 \le 4$ and $0 \le z \le 3$.

(HINT: Use Gauss' Theorem.)

- **a.** 24π
- **b.** 48π
- **c.** 96π correctchoice
- **d.** 144π
- **e.** 324π

$$\iint\limits_{\partial C} \vec{F}\bullet d\vec{S} = \iiint\limits_{C} \nabla\bullet F\,dV = \iiint\limits_{C} (3x^2 + 3y^2 + x^2 + y^2)\,dx\,dy\,dz$$

$$= \int_0^3\int_0^{2\pi}\int_0^2 4r^2\,r\,dr\,d\theta\,dz = 3\bullet 2\pi\bullet [r^4]_0^2 = 96\pi$$

6. (45 points) Consider the vector space V spanned by

$$e_1 = \cosh^2 x, \quad e_2 = \sinh^2 x \quad \text{and} \quad e_3 = \cosh x \sinh x.$$

USEFUL FACTS:

$\sinh 0 = 0$	$\cosh^2 x - \sinh^2 x = 1$	$\frac{d}{dx}\sinh^2 x = 2\cosh x \sinh x$
$\cosh 0 = 1$		
$\frac{d}{dx}\sinh x = \cosh x$	$\sinh(2x) = 2\sinh x \cosh x$	$\frac{d}{dx}\cosh^2 x = 2\cosh x \sinh x$
$\frac{d}{dx}\cosh x = \sinh x$	$\cosh(2x) = \cosh^2 x + \sinh^2 x$	$\frac{d}{dx}\cosh x \sinh x = \cosh^2 x + \sinh^2 x$

a. (5 pts Extra Credit) Show e_1, e_2 and e_3 are linearly independent.

Assume $\quad ae_1 + be_2 + ce_3 = 0.$

Then $\quad a\cosh^2 x + b\sinh^2 x + c\cosh x \sinh x = 0$ for all x. (1)

Plug $x = 0$ into (1) $\quad a(1) + b(0) + c(0) = 0 \quad \Rightarrow \quad a = 0$

With $a = 0$ differentiate (1) $\quad b2\cosh x \sinh x + c\left(\cosh^2 x + \sinh^2 x\right) = 0$ for all x. (2)

Plug $x = 0$ into (2) $\quad b(0) + c(1) = 0 \quad \Rightarrow \quad c = 0$

With $c = 0$ differentiate (2) $\quad b2\left(\cosh^2 x + \sinh^2 x\right) = 0$ for all x. (3)

Plug $x = 0$ into (3) $\quad b2(1) = 0 \quad \Rightarrow \quad b = 0$

Therefore $a = b = c = 0$ and so e_1, e_2 and e_3 are linearly independent.

b. (10 pts) Another basisfor V is $f_1 = 1, \quad f_2 = \cosh(2x)$ and $f_3 = \sinh(2x)$. Find the change of basis matrices from the *f*-basis to the e-basis, $\underset{e \leftarrow f}{C}$ and from the e-basis to the *f*-basis, $\underset{f \leftarrow e}{C}$.

Since

$$f_1 = 1 = \cosh^2 x - \sinh^2 x = e_1 - e_2$$
$$f_2 = \cosh(2x) = \cosh^2 x + \sinh^2 x = e_1 + e_2$$
$$f_3 = \sinh(2x) = 2\sinh x \cosh x = 2e_3$$

the change of basis matrix is $\underset{e \leftarrow f}{C} = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

Invert to find $\underset{f \leftarrow e}{C}$

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 & \frac{1}{2} & \frac{-1}{2} & 0 \\ 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} \end{pmatrix} \Rightarrow \underset{f \leftarrow e}{C} = \begin{pmatrix} \frac{1}{2} & \frac{-1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

c. (5 pts) Consider the linear operator of differentiation

$$D : V \to V \quad \text{given by} \quad D(g) = \frac{dg}{dx}$$

Since

$$D(f_1) = D(1) = 0$$
$$D(f_2) = D(\cosh(2x)) = 2\sinh(2x) = 2e_3$$
$$D(f_3) = D(\sinh(2x)) = 2\cosh(2x) = 2e_2$$

the matrix of D relative to the f-basis is $\underset{f\leftarrow f}{B} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix}$.

Find the matrix of D relative to the e-basis. Call it $\underset{e\leftarrow e}{A}$.

$$\underset{e\leftarrow e}{A} = \underset{e\leftarrow f}{C} \; \underset{f\leftarrow f}{B} \; \underset{f\leftarrow e}{C} == \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{-1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 2 & 2 & 0 \end{pmatrix}$$

d. (5 pts) Find the matrix of D relative to the e-basis by a second method.

Since

$$D(e_1) = D\left(\cosh^2 x\right) = 2\cosh x \sinh x = 2e_3$$
$$D(e_2) = D\left(\sinh^2 x\right) = 2\cosh x \sinh x = 2e_3$$
$$D(e_3) = D(\cosh x \sinh x) = \cosh^2 x + \sinh^2 x = e_1 + e_2$$

the matrix of D relative to the e-basis is $\underset{e\leftarrow e}{A} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 2 & 2 & 0 \end{pmatrix}$.

e. (16 pts) Find the eigenvalues and eigenvectors of the matrix $B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix}$.

$$\det(B - \lambda\mathbf{1}) = \det\begin{pmatrix} -\lambda & 0 & 0 \\ 0 & -\lambda & 2 \\ 0 & 2 & -\lambda \end{pmatrix} = -\lambda(\lambda^2 - 4) = -\lambda(\lambda - 2)(\lambda + 2) = 0$$

Eigenvalues: $\lambda = 0, 2, -2$

For $\lambda_1 = 0$: $\quad (B - \lambda\mathbf{1})\vec{v}_1 == \vec{0} \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix}\vec{v}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} t \\ 0 \\ 0 \end{pmatrix} \Rightarrow \vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

For $\lambda_2 = 2$: $\quad (B - \lambda\mathbf{1})\vec{v}_2 == \vec{0} \quad \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 2 \\ 0 & 2 & -2 \end{pmatrix}\vec{v}_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 2 & -2 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ t \\ t \end{pmatrix} \Rightarrow \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

For $\lambda_3 = -2$: $\quad (B - \lambda\mathbf{1})\vec{v}_3 == \vec{0} \quad \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 2 \end{pmatrix}\vec{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -t \\ t \end{pmatrix} \Rightarrow \vec{v}_3 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

f. (4 pts) Find the eigenvalues and eigenfunctions of the operator D.

For $\lambda_1 = 0$: $\quad \vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad g_1 = 1f_1 = 1$

For $\lambda_2 = 2$: $\quad \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \quad g_2 = 1f_2 + 1f_3 = \cosh 2x + \sinh 2x = e^{2x}$

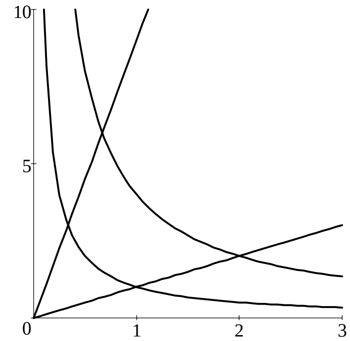
For $\lambda_3 = -2$: $\quad \vec{v}_3 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \quad g_3 = -1f_2 + 1f_3 = -\cosh 2x + \sinh 2x = e^{-2x}$

7. (11 points) Compute $\iint \frac{1}{x} e^{\sqrt{xy}}\, dx\, dy$ over the diamond shaped region between the curves

$$y = x, \qquad y = 9x, \qquad y = \frac{1}{x} \qquad \text{and} \qquad y = \frac{4}{x}.$$

You **must** use the curvilinear coordinates

$$x = \frac{v}{u} \qquad \text{and} \qquad y = uv.$$



a. (3 pts) Find the Jacobian:

$$J = \left| \det \begin{pmatrix} \frac{-v}{u^2} & \frac{1}{u} \\ v & u \end{pmatrix} \right| = \left| \frac{-2v}{u} \right| = \frac{2v}{u}$$

b. (4 pts) Express each boundary curve in terms of u and v:

$$y = x \quad \Rightarrow \quad uv = \frac{v}{u} \quad \Rightarrow \quad u = 1$$
$$y = 9x \quad \Rightarrow \quad uv = 9\frac{v}{u} \quad \Rightarrow \quad u = 3$$
$$y = \frac{1}{x} \quad \Rightarrow \quad uv = \frac{u}{v} \quad \Rightarrow \quad v = 1$$
$$y = \frac{4}{x} \quad \Rightarrow \quad uv = 4\frac{u}{v} \quad \Rightarrow \quad v = 2$$

c. (2 pts) Express the integrand in terms of u and v:

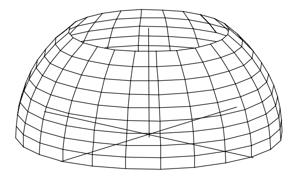
$$xy = v^2 \qquad \sqrt{xy} = v, \qquad \frac{1}{x} = \frac{u}{v}, \qquad \frac{1}{x} e^{\sqrt{xy}} = \frac{u}{v} e^v$$

d. (2 pts) Compute the integral:

$$\iint \frac{1}{x} e^{\sqrt{xy}}\, dx\, dy = \int_1^2 \int_1^3 \frac{u}{v} e^v \frac{2v}{u}\, du\, dv = 2\int_1^2 \int_1^3 e^v\, du\, dv = 2\Big[u \Big]_1^2 \Big[e^v \Big]_1^3 = 2(e^3 - e)$$

8. (24 points) Use two methods to compute

$$\iint_S \vec{\nabla} \times \vec{F} \bullet d\vec{S}$$



for $\vec{F} = (y, -x, z^2)$ over the piece of the sphere $x^2 + y^2 + z^2 = 25$ for $0 \le z \le 4$ with normal pointing away from the z-axis.

a. (12 pts) Parametrize the surface, compute $\vec{\nabla} \times \vec{F}$ and compute the double integral $\iint_S \vec{\nabla} \times \vec{F} \bullet d\vec{S}$ directly.

$\vec{R}(\varphi, \theta) = (5\sin\varphi\cos\theta, 5\sin\varphi\sin\theta, 5\cos\varphi)$

$\vec{R}_\varphi = (5\cos\varphi\cos\theta, 5\cos\varphi\sin\theta, -5\sin\varphi)$

$\vec{R}_\theta = (-5\sin\varphi\sin\theta, 5\sin\varphi\cos\theta, 0)$

$\vec{N} = \langle 25\sin^2\varphi\cos\theta, 25\sin^2\varphi\sin\theta, 25\sin\varphi\cos\varphi \rangle$ which points away from the z-axis.

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ y & -x & z^2 \end{vmatrix} = (0, 0, -2)$$

On the upper circle, $z = 4$, $x^2 + y^2 = 9$, $\tan\varphi = \frac{3}{4}$, $\varphi = \tan^{-1}\left(\frac{3}{4}\right)$. On the lower circle, $\varphi = \frac{\pi}{2}$

$$\iint_S \vec{\nabla} \times \vec{F} \bullet d\vec{S} = \iint_S \vec{\nabla} \times \vec{F} \bullet \vec{N}\, d\varphi\, d\theta = -50 \int_0^{2\pi} \int_{\tan^{-1}(3/4)}^{\pi/2} \sin\varphi\cos\varphi\, d\varphi\, d\theta$$

$$= -50[2\pi]\left[\frac{\sin^2\varphi}{2}\right]_{\tan^{-1}(3/4)}^{\pi/2} = -50\pi\left[1 - \left(\frac{3}{5}\right)^2\right] = -\pi[50 - 18] = -32\pi$$

b. (12 pts) By Stokes' Theorem $\iint_S \vec{\nabla} \times \vec{F} \bullet d\vec{S} = \oint_{\partial S} \vec{F} \bullet d\vec{s}$ where ∂S is the boundary of S.

Parametrize the upper and lower circles and compute $\oint \vec{F} \bullet d\vec{s}$ for each circle.

Be sure to discuss the orientation of the circles when you add up the integrals.

On the upper circle, $z = 4$, $x^2 + y^2 = 9$, $\vec{r}(t) = (3\cos t, 3\sin t, 4)$, $\vec{v} = (-3\sin t, 3\cos t, 0)$

$\vec{F} = (y, -x, z^2) = (3\sin t, -3\cos t, 16)$, $\vec{F} \bullet \vec{v} = -9\sin^2 t - 9\cos^2 t = -9$

$$\oint_{\text{upper}} \vec{F} \bullet d\vec{s} = \int_0^{2\pi} (-9)\, dt = -18\pi$$

On the lower circle, $z = 0$, $x^2 + y^2 = 25$, $\vec{r}(t) = (5\cos t, 5\sin t, 0)$, $\vec{v} = (-5\sin t, 5\cos t, 0)$

$\vec{F} = (y, -x, z^2) = (5\sin t, -5\cos t, 0)$, $\vec{F} \bullet \vec{v} = -25\sin^2 t - 16\cos^2 t = -25$

$$\oint_{\text{lower}} \vec{F} \bullet d\vec{s} = \int_0^{2\pi} (-25)\, dt = -50\pi$$

Both the upper and lower integrals were computed counterclockwise. We need the upper integral clockwise. So it gets a minus sign.

$$\oint_{\partial S} \vec{F} \bullet d\vec{s} = \oint_{\text{lower}} \vec{F} \bullet d\vec{s} - \oint_{\text{upper}} \vec{F} \bullet d\vec{s} = -50\pi + 18\pi = -32\pi$$