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Math 311 Exam 1 Spring 2002

Section 503 P. Yasskin

Solutions

1	/10
2	/10
3	/30
4	/25
5	/25

1. $\left(10 \text{ points}\right)$ A matrix A satisfies $E_3 E_2 E_1 A = U$ where

$$E_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} \qquad U = \begin{pmatrix} 2 & 5 & * \\ 0 & -3 & * \\ 0 & 0 & -1 \end{pmatrix}$$

and the $*$'s represent unknown non-zero numbers. Find $\det A$.

$$\det E_1 = -1 \qquad \det E_2 = \frac{1}{2} \qquad \det E_3 = 1 \qquad \det U = (2)(-3)(-1) = 6$$

$$\det A = \frac{\det U}{\det E_1 \det E_2 \det E_3} = \frac{6}{(-1)\left(\frac{1}{2}\right)(1)} = -12$$

2. $\left(10 \text{ points}\right)$ If c is a scalar, A is a 50×60 matrix and B is a 60×80 matrix, prove $A(cB) = c(AB)$.
HINT: Write out the ij-component of each side.

$$[A(cB)]_{ij} = \sum_{k=1}^{60} A_{ik}(cB)_{kj} = \sum_{k=1}^{60} A_{ik} c B_{kj} = c \sum_{k=1}^{60} A_{ik} B_{kj}$$

$$[c(AB)]_{ij} = c(AB)_{ij} = c \sum_{k=1}^{60} A_{ik} B_{kj}$$

These are equal. So $A(cB) = c(AB)$

3. (30 points) Consider the triangle with vertices

$$A = (2,4,0) \qquad B = (4,2,1) \qquad C = (2,7,4)$$

a. Find $\cos\theta$ where θ is the angle at vertex A.

$$\overrightarrow{AB} = B - A = (2,-2,1) \qquad \overrightarrow{AC} = C - A = (0,3,4)$$

$$\cos\theta = \frac{\overrightarrow{AB} \cdot \overrightarrow{AC}}{\left|\overrightarrow{AB}\right| \left|\overrightarrow{AC}\right|} = \frac{-6+4}{\sqrt{4+4+1}\,\sqrt{9+16}} = \frac{-2}{15}$$

b. Find the area of the triangle ΔABC.

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} i & j & k \\ 2 & -2 & 1 \\ 0 & 3 & 4 \end{vmatrix} = i(-8-3) - j(8) + k(6) = (-11,-8,6)$$

$$Area = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \frac{1}{2}\sqrt{121+64+36} = \frac{1}{2}\sqrt{221}$$

c. Find a set of parametric equations for the line containing A and C.

$$X = A + t\,\overrightarrow{AC} \qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix} \qquad \begin{aligned} x &= 2 \\ y &= 4 + 3t \\ z &= 4t \end{aligned}$$

d. Find a set of parametric equations for the plane containing A, B and C.

$$X = A + s\,\overrightarrow{AB} + t\,\overrightarrow{AC} \qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix} + s\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} + t\begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix} \qquad \begin{aligned} x &= 2 + 2s \\ y &= 4 - 2s + 3t \\ z &= s + 4t \end{aligned}$$

e. Find a non-parametric equation for the plane containing A, B and C.

$$\vec{N} = \overrightarrow{AB} \times \overrightarrow{AC} = (-11,-8,6) \qquad \vec{N} \cdot X = \vec{N} \cdot A$$

$$-11x - 8y + 6z = -11 \cdot 2 - 8 \cdot 4 + 6 \cdot 0 = -54 \qquad 11x + 8y - 6z = 54$$

4. (25 points) Consider the system of equations:

$$AX = B \quad \text{where} \quad A = \begin{pmatrix} 2 & 0 & 1 \\ 1 & -2 & 1 \\ 0 & 3 & -1 \end{pmatrix} \quad X = \begin{pmatrix} x & p \\ y & q \\ z & r \end{pmatrix} \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -1 \end{pmatrix}$$

Compute A^{-1}. (Give reasons for each step.)

$$\left(\begin{array}{ccc|ccc} 2 & 0 & 1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 & 1 & 0 \\ 0 & 3 & -1 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} R_2 \\ R_1 \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & -2 & 1 & 0 & 1 & 0 \\ 2 & 0 & 1 & 1 & 0 & 0 \\ 0 & 3 & -1 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \\ R_2 - 2R_1 \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & -2 & 1 & 0 & 1 & 0 \\ 0 & 4 & -1 & 1 & -2 & 0 \\ 0 & 3 & -1 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \\ R_2 - R_3 \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & -2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & -2 & -1 \\ 0 & 3 & -1 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} R_1 + 2R_2 \\ \\ R_3 - 3R_2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 2 & -3 & -2 \\ 0 & 1 & 0 & 1 & -2 & -1 \\ 0 & 0 & -1 & -3 & 6 & 4 \end{array}\right) \begin{array}{l} R_1 + R_3 \\ \\ -R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 3 & 2 \\ 0 & 1 & 0 & 1 & -2 & -1 \\ 0 & 0 & 1 & 3 & -6 & -4 \end{array}\right) \qquad A^{-1} = \begin{pmatrix} -1 & 3 & 2 \\ 1 & -2 & -1 \\ 3 & -6 & -4 \end{pmatrix}$$

Solve $AX = B$.

$$X = A^{-1}B = \begin{pmatrix} -1 & 3 & 2 \\ 1 & -2 & -1 \\ 3 & -6 & -4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \\ -1 & -2 \end{pmatrix}$$

5. (25 points) Consider the system of equations:

$$\begin{aligned} 3w + 6x + y \quad &= 5 \\ y - 3z &= 2 \\ w + 2x + y - 2z &= 3 \\ -2w - 4x + y - 5z &= b \end{aligned}$$

Find the value(s) of b for which there exist solutions. (Give reasons for each step.)

$$\left(\begin{array}{cccc|c} 3 & 6 & 1 & 0 & 5 \\ 0 & 0 & 1 & -3 & 2 \\ 1 & 2 & 1 & -2 & 3 \\ -2 & -4 & 1 & -5 & b \end{array}\right) \begin{array}{l} R_3 \\ \\ R_1 \\ \\ \end{array}$$

$$\left(\begin{array}{cccc|c} 1 & 2 & 1 & -2 & 3 \\ 0 & 0 & 1 & -3 & 2 \\ 3 & 6 & 1 & 0 & 5 \\ -2 & -4 & 1 & -5 & b \end{array}\right) \begin{array}{l} \\ \\ R_3 - 3R_1 \\ R_4 + 2R_1 \end{array}$$

$$\left(\begin{array}{cccc|c} 1 & 2 & 1 & -2 & 3 \\ 0 & 0 & 1 & -3 & 2 \\ 0 & 0 & -2 & 6 & -4 \\ 0 & 0 & 3 & -9 & b+6 \end{array}\right) \begin{array}{l} R_1 - R_2 \\ \\ R_3 + 2R_2 \\ R_4 - 3R_2 \end{array}$$

$$\left(\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & -3 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b \end{array}\right)$$

To have solutions we must have $b = 0$.

For that value (those values) of b what is the solution set?

$w = 1 - 2s - t$

$x = s$

$y = 2 + 3t$

$z = t$

Give a geometrical description of the solution set.

Plane in $\mathbf{R}^4$.