Math 311 Exam 3 Spring 2002

Section 503 Solutions P. Yasskin

Multiple Choice (8 points each.)

1. If $F = (xy, yz, xz)$ then $\vec{\nabla} \cdot \vec{F} =$

- **a.** $y - z + x$
- **b.** $(-y, z, -x)$
- **c.** $x + y + z$ CORRECT
- **d.** $(-y, -z, -x)$
- **e.** $-x + y - z$

$$\vec{\nabla} \cdot \vec{F} = \partial_x(xy) + \partial_y(yz) + \partial_z(xz) = y + z + x$$

2. If $F = (xy, yz, xz)$ then $\vec{\nabla} \times \vec{F} =$

- **a.** $y - z + x$
- **b.** $(-y, z, -x)$
- **c.** $x + y + z$
- **d.** $(-y, -z, -x)$ CORRECT
- **e.** $-x + y - z$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ xy & yz & xz \end{vmatrix} = \hat{i}(0 - y) - \hat{j}(z - 0) + \hat{k}(0 - x) = (-y, -z, -x)$$

3. If $f(x,y,z) = x\sin(yz) - y\cos(xz) + z\tan(xy)$ then $\vec{\nabla} \times \vec{\nabla} f =$

- **a.** $z\sin(yz)\vec{i} + z\cos(xz)\vec{j} + xy\sec^2(xy)\vec{k}$
- **b.** $\sin(yz)\vec{i} - \cos(xz)\vec{j} + \tan(xy)\vec{k}$
- **c.** $\cos(yz)\vec{i} + \sin(xz)\vec{j} + \sec^2(xy)\vec{k}$
- **d.** 0 CORRECT
- **e.** Does not exist.

$\vec{\nabla} \times \vec{\nabla} f == 0$ for any twice differentiable function f.

4. Compute the line integral $\int y\,dx - x\,dy$ counterclockwise around the semicircle $x^2 + y^2 = 4$ from $(2,0)$ to $(-2,0)$. (HINT: Parametrize the curve.)

- **a.** -4π CORRECT
- **b.** -2π
- **c.** π
- **d.** 2π
- **e.** 4π

$\vec{r}(\theta) = (2\cos\theta, 2\sin\theta)$ $\vec{v}(\theta) = (-2\sin\theta, 2\cos\theta)$

$\vec{F}(x,y) = (y, -x)$ $\vec{F}(\vec{r}(\theta)) = (2\sin\theta, -2\cos\theta)$

$$\int y\,dx - x\,dy = \int \vec{F} \cdot d\vec{s} = \int_0^\pi \vec{F}(\vec{r}(\theta)) \cdot \vec{v}(\theta)\,d\theta = \int_0^\pi -4(\sin^2\theta + \cos^2\theta)\,d\theta = -4\pi$$

5. Compute the line integral $\int \vec{F} \cdot d\vec{s}$ for the vector field $\vec{F} = \left(\frac{1}{x}, \frac{1}{y}\right)$ along the curve $\vec{r}(t) = \left(e^{\cos(t^2)}, e^{\sin(t^2)}\right)$ for $0 \le t \le \sqrt{\pi}$. (HINT: Find a potential f so that $F = \vec{\nabla} f$.)

a. -2 CORRECT

b. 0

c. $\frac{2}{e}$

d. 1

e. π

$$F = \vec{\nabla} f \qquad \left(\frac{1}{x}, \frac{1}{y}\right) = \left(\frac{df}{dx}, \frac{df}{dy}\right) \qquad f = \ln x + \ln y$$

$$A = \vec{r}(0) = (e^{\cos(0)}, e^{\sin(0)}) = (e, 1) \qquad B = \vec{r}(\sqrt{\pi}) = (e^{\cos(\pi)}, e^{\sin(\pi)}) = (e^{-1}, 1)$$

$$\int \vec{F} \cdot d\vec{s} = \int_A^B \vec{\nabla} f \cdot d\vec{s} = f(B) - f(A) = f((e^{-1}, 1)) - f((e, 1)) = (\ln e^{-1} + \ln 1) - (\ln e + \ln 1) = -2$$

6. Compute $\oint (5x + 3y)\,dx + (x - 2y)\,dy$ counterclockwise around the edge of the rectangle $1 \le x \le 5$, $3 \le y \le 6$. (HINT: Use Green's Theorem.)

a. 36

b. 24

c. 12

d. -24 CORRECT

e. -36

$$P = (5x + 3y) \qquad Q = (x - 2y)$$

$$\oint_{\partial R} P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \int_3^6 \int_1^5 (1 - 3)\; dx\,dy = -2(5 - 1)(6 - 3) = -24$$

7. Compute $\iint_{\partial C} \vec{F} \cdot d\vec{S}$ for the vector field $\vec{F} = \left(zx^3, zy^3, z^2(x^2 + y^2)\right)$ over the total surface of the solid cylinder $C = \left\{(x,y,z) \mid x^2 + y^2 \le 4,\ 0 \le z \le 3\right\}$ with outward normal. (HINT: Use Gauss' Theorem.)

a. 360π

b. 180π CORRECT

c. 90π

d. 60π

e. 30π

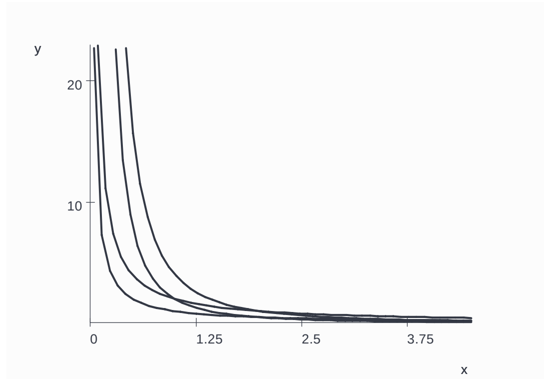
$$\vec{\nabla} \cdot \vec{F} = 3zx^2 + 3zy^2 + 2z(x^2 + y^2) = 5z(x^2 + y^2) = 5zr^2$$

$$\iint_{\partial C} \vec{F} \cdot d\vec{S} = \iiint_C \vec{\nabla} \cdot \vec{F}\,dV = \int_0^3 \int_0^{2\pi} \int_0^2 5zr^2\,r\,dr\,d\theta\,dz = 5\left[\frac{z^2}{2}\right]_0^3 [2\pi] \left[\frac{r^4}{4}\right]_0^2 = 5\left(\frac{9}{2}\right) 2\pi(4) = 180\pi$$

8. (20 points) Compute $\iint\limits_R x^2 y \; dx\, dy$ over the diamond shaped region R bounded by

$$y = \frac{1}{x}, \qquad y = \frac{2}{x}, \qquad y = \frac{2}{x^2}, \qquad y = \frac{4}{x^2}$$

For full credit you must use curvilinear coordinates. Half credit for rectangular coordinates.



$$\left.\begin{array}{l} u = xy \\ v = x^2 y \end{array}\right\} \Rightarrow \text{ Boundaries are } \begin{array}{ll} u = 1 & u = 2 \\ v = 2 & v = 4 \end{array}$$

Solve for x and y:

$$\left.\begin{array}{l} \dfrac{v}{u} = \dfrac{x^2 y}{xy} = x \\[2ex] \dfrac{u^2}{v} = \dfrac{x^2 y^2}{x^2 y} = y \end{array}\right\} \Rightarrow \left.\begin{array}{l} x = \dfrac{v}{u} \\[2ex] y = \dfrac{u^2}{v} \end{array}\right\} \Rightarrow$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\[2ex] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} -\dfrac{v}{u^2} & \dfrac{1}{u} \\[2ex] \dfrac{2u}{v} & -\dfrac{u^2}{v^2} \end{vmatrix} = \left(-\frac{v}{u^2}\right)\left(-\frac{u^2}{v^2}\right) - \left(\frac{1}{u}\right)\left(\frac{2u}{v}\right) = \frac{1}{v} - \frac{2}{v} = -\frac{1}{v}$$

Jacobian: $J = \left|\dfrac{\partial(x,y)}{\partial(u,v)}\right| = \dfrac{1}{v}$ Integrand: $x^2 y = v$

$$\iint\limits_R x^2 y \; dx\, dy = \int_2^4 \int_1^2 v \, \frac{1}{v} \; du\, dv = (4-2)(2-1) = 2$$

9. (24 points) Stokes' Theorem states that if S is a nice surface in $\mathbf{R}^3$ and ∂S is its boundary curve traversed counterclockwise as seen from the tip of the normal to S and $\vec{F}$ is a nice vector field on S then

$$\iint_S \vec{\nabla} \times \vec{F} \cdot d\vec{S} = \oint_{\partial S} \vec{F} \cdot d\vec{s}$$

Verify Stokes' Theorem if $F = (-yx^2, xy^2, x^2 + y^2)$ and S is the part of the cone $z = \sqrt{x^2 + y^2}$ below $z = 2$ with normal pointing up and in.

9a. (4 points) Compute $\vec{\nabla} \times \vec{F}$. (HINT: Use rectangular coordinates.)

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ -yx^2 & xy^2 & x^2 + y^2 \end{vmatrix} = \hat{i}(2y - 0) - \hat{j}(2x - 0) + \hat{k}(y^2 - -x^2) = (2y, -2x, x^2 + y^2)$$

9b. (10 points) Compute $\iint_S \vec{\nabla} \times \vec{F} \cdot d\vec{S}$.

(HINT: Here is the parametrization of the cone and the steps you should use. Remember to check the orientation of the surface.)

$$\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, r)$$

$$\vec{R}_r = (\cos\theta, \sin\theta, 1)$$

$$\vec{R}_\theta = (-r\sin\theta, r\cos\theta, 0)$$

$$\vec{N} = \hat{i}(-r\cos\theta) - \hat{j}(r\sin\theta) + \hat{k}(r) = (-r\cos\theta, -r\sin\theta, r)$$ Points up and in.

$$\left(\vec{\nabla} \times \vec{F}\right)\left(\vec{R}(r,\theta)\right) = (2r\sin\theta, -2r\cos\theta, r^2)$$

$$\left(\vec{\nabla} \times \vec{F}\right) \cdot \vec{N} = -2r^2 \sin\theta\cos\theta + 2r^2\cos\theta\sin\theta + r^3 = r^3$$

$$\iint_S \vec{\nabla} \times \vec{F} \cdot d\vec{S} = \int_0^{2\pi}\int_0^2 r^3\, dr\, d\theta = 2\pi\left[\frac{r^4}{4}\right]_0^2 = 8\pi$$

9c. (10 points) Compute $\oint_{\partial S} \vec{F} \cdot d\vec{s}$. Recall $F = (-yx^2, xy^2, x^2 + y^2)$.

(HINT: Parametrize of the boundary circle. Remember to check the orientation of the curve.)
(CHECK the answers to 9b and 9c agree.)

$$\vec{r}(\theta) = (2\cos\theta, 2\sin\theta, 2)$$

$$\vec{v}(\theta) = (-2\sin\theta, 2\cos\theta, 0)$$ Counterclockwise from +z-axis.

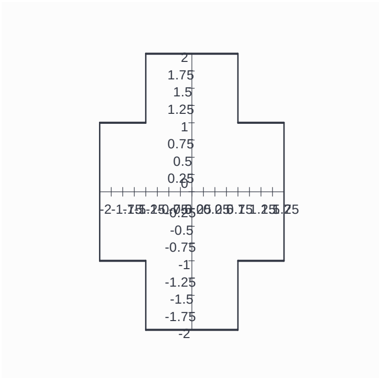
$$\vec{F}(\vec{r}(\theta)) = (-2\sin\theta \cdot 4\cos^2\theta, 2\cos\theta \cdot 4\sin^2\theta, 4\cos^2\theta + 4\sin^2\theta) = (-8\sin\theta\cos^2\theta, 8\cos\theta\sin^2\theta, 4)$$

$$\vec{F} \cdot \vec{v} = 16\sin^2\theta\cos^2\theta + 16\cos^2\theta\sin^2\theta = 32\sin^2\theta\cos^2\theta$$

$$\oint_{\partial S} \vec{F} \cdot d\vec{s} = \int_0^{2\pi} 32\sin^2\theta\cos^2\theta\, d\theta = \int_0^{2\pi} 8\sin^2(2\theta)\, d\theta = 8 \cdot \frac{1}{2}(2\pi - 0) = 8\pi$$

10. (10 points Extra Credit) Compute the line integral $\oint \frac{y}{x^2+y^2}\,dx - \frac{x}{x^2+y^2}\,dy$ counterclockwise around the boundary of the plus sign shown below.

Be sure to justify any theorem you use. (Hint: The answer is not zero.)



$P = \frac{y}{x^2+y^2}$ $\qquad Q = \frac{-x}{x^2+y^2}$ $\qquad$ P and Q are defined everywhere except the origin.

$$\partial_x Q - \partial_y P = \frac{(x^2+y^2)(-1)-(-x)2x}{(x^2+y^2)^2} - \frac{(x^2+y^2)(1)-(y)2y}{(x^2+y^2)^2} = \frac{-x^2-y^2+2x^2-x^2-y^2+2y^2}{(x^2+y^2)^2} = 0$$

By Green's Theorem, the path may be continuously deformed without changing the value of the integral provided the region between the curves does not contain the origin. So integrate counterclockwise around the unit circle.

$x = \cos\theta, \qquad y = \sin\theta, \qquad dx = -\sin\theta\,d\theta, \qquad dy = \cos\theta\,d\theta$

Note: $x^2+y^2 = 1$ $\qquad$ So $\qquad P = y = \sin\theta \qquad Q = -x = -\cos\theta$

$$\oint \frac{y}{x^2+y^2}\,dx - \frac{x}{x^2+y^2}\,dy = \oint \sin\theta\,(-\sin\theta)\,d\theta - \cos\theta\,\cos\theta\,d\theta = -\int_0^{2\pi} d\theta = -2\pi$$