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Math 311 Final Exam Spring 2002

Section 503 P. Yasskin

Solutions

1 EC	/10	3	/30
3	/40	4	/30

1. (10 points Extra Credit) Determine if and where the line $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + t\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$

intersects the plane $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -8 \\ 3 \end{pmatrix} + r\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + s\begin{pmatrix} 3 \\ 6 \\ 4 \end{pmatrix}$.

Equate x, y and z to get the equations

$$\begin{aligned} 1 + 4t &= 2 + r + 3s \\ 2 + 5t &= -8 - r + 6s \\ 3 + 6t &= 3 + 2r + 4s \end{aligned} \qquad \text{or} \qquad \begin{aligned} r + 3s - 4t &= -1 \\ -r + 6s - 5t &= 10 \\ 2r + 4s - 6t &= 0 \end{aligned}$$

Solve for r, s and t:

$$\left(\begin{array}{ccc|c} 1 & 3 & -4 & -1 \\ -1 & 6 & -5 & 10 \\ 2 & 4 & -6 & 0 \end{array}\right) \begin{array}{c} \\ R_2 + R_1 \\ R_3 - 2R_1 \end{array} \Rightarrow \left(\begin{array}{ccc|c} 1 & 3 & -4 & -1 \\ 0 & 9 & -9 & 9 \\ 0 & -2 & 2 & 2 \end{array}\right) \begin{array}{c} \\ \frac{1}{9}R_2 \\ -\frac{1}{2}R_3 \end{array}$$

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & 3 & -4 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & -1 \end{array}\right) \begin{array}{c} \\ \\ R_3 - R_2 \end{array} \Rightarrow \left(\begin{array}{ccc|c} 1 & 3 & -4 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & -2 \end{array}\right)$$

$\Rightarrow$ $0 = -2$ $\Rightarrow$ Contradiction $\Rightarrow$ No solutions

$\Rightarrow$ The line does not intersect the plane.

2. (40 points) Let $M(p,q)$ be the vector space of $p \times q$ matrices, and let $P = \begin{pmatrix} 2 & 1 \\ 4 & 2 \\ 6 & 3 \end{pmatrix}$.

Consider the linear function

$$L : M(2,2) \to M(3,2) \quad \text{given by} \quad L(X) = PX$$

a. (5) Let $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and compute $L(X)$.

$$L\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 4 & 2 \\ 6 & 3 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 2a+c & 2b+d \\ 4a+2c & 4b+2d \\ 6a+3c & 6b+3d \end{pmatrix}$$

b. (2) Identify the domain of L. What is its dimension?

$Dom(L) = M(2,2) \qquad \dim Dom(L) = 4$

c. (2) Identify the codomain of L. What is its dimension?

$Codom(L) = M(3,2) \qquad \dim Codom(L) = 6$

d. (8) Identify the kernel (null space) of L. Give a basis and the dimension.

$$L(X) = \mathbf{0} \quad \Rightarrow \quad \begin{matrix} 2a+c=0 & 2b+d=0 \\ 4a+2c=0 & 4b+2d=0 \\ 6a+3c=0 & 6b+3d=0 \end{matrix} \quad \Rightarrow \quad \begin{matrix} c=-2a \\ d=-2b \end{matrix}$$

$$\Rightarrow \quad X = \begin{pmatrix} a & b \\ -2a & -2b \end{pmatrix}$$

$$Ker(L) = \left\{\begin{pmatrix} a & b \\ -2a & -2b \end{pmatrix}\right\} = \left\{a\begin{pmatrix} 1 & 0 \\ -2 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & -2 \end{pmatrix}\right\}$$

$$= Span\left\{\begin{pmatrix} 1 & 0 \\ -2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & -2 \end{pmatrix}\right\}$$

Basis is $\left\{\begin{pmatrix} 1 & 0 \\ -2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & -2 \end{pmatrix}\right\}$. $\quad \dim Ker(L) = 2$

Problem **2** continued:

e. (8) Identify the image (range) of L. Give a basis and the dimension.

$$Im(L) = \left\{ \begin{pmatrix} 2a+c & 2b+d \\ 4a+2c & 4b+2d \\ 6a+3c & 6b+3d \end{pmatrix} \right\}$$

$$= \left\{ a\begin{pmatrix} 2 & 0 \\ 4 & 0 \\ 6 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 2 \\ 0 & 4 \\ 0 & 6 \end{pmatrix} + c\begin{pmatrix} 1 & 0 \\ 2 & 0 \\ 3 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 1 \\ 0 & 2 \\ 0 & 3 \end{pmatrix} \right\}$$

$$= Span\left\{ \begin{pmatrix} 2 & 0 \\ 4 & 0 \\ 6 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 0 & 4 \\ 0 & 6 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 0 \\ 3 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 2 \\ 0 & 3 \end{pmatrix} \right\}$$

(Not Linearly Independent!)

$$= Span\left\{ \begin{pmatrix} 1 & 0 \\ 2 & 0 \\ 3 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 2 \\ 0 & 3 \end{pmatrix} \right\}$$

Basis is $\left\{ \begin{pmatrix} 1 & 0 \\ 2 & 0 \\ 3 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 2 \\ 0 & 3 \end{pmatrix} \right\}$. $\quad \dim Im(L) = 2$

f. (3) Is L onto? Why?

L is NOT onto because $\dim Im(L) = 2$ but $\dim Codom(L) = 6$, so $Im(L) \neq Codom(L)$.

g. (3) Is L one-to-one? Why?

L is NOT one-to-one because $Ker(L) \neq \{\mathbf{0}\}$.

h. (2) Check that the dimensions of the kernel and image are consistent with the dimensions of the domain and codomain.

$\dim Ker(L) + \dim Im(L) = 2 + 2 = 4 = \dim Dom(L)$

Problem **2** continued:

i. (7) Find the matrix of L relative to the bases

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad E_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad E_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$F_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \quad F_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} \quad F_3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$F_4 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \quad F_5 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad F_6 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Recall: $$L\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 2a+c & 2b+d \\ 4a+2c & 4b+2d \\ 6a+3c & 6b+3d \end{pmatrix}$$

$$L(E_1) = L\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 4 & 0 \\ 6 & 0 \end{pmatrix} = 2F_1 + 4F_2 + 6F_3$$

$$L(E_2) = L\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 0 & 4 \\ 0 & 6 \end{pmatrix} = 2F_4 + 4F_5 + 6F_6$$

$$L(E_3) = L\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 0 \\ 3 & 0 \end{pmatrix} = 1F_1 + 2F_2 + 3F_3$$

$$L(E_4) = L\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 2 \\ 0 & 3 \end{pmatrix} = 1F_4 + 2F_5 + 3F_6$$

$$A = \begin{pmatrix} 2 & 0 & 1 & 0 \\ 4 & 0 & 2 & 0 \\ 6 & 0 & 3 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 4 & 0 & 2 \\ 0 & 6 & 0 & 3 \end{pmatrix}$$

3. $\left(30 \text{ points}\right)$ On the vector space $P_2 = \left\{\text{polynomials of degree less than } 2\right\}$ consider the function of two polynomials given by

$$\langle p,q\rangle = \int_0^\infty p(x)q(x)e^{-x}\,dx$$

a. (15) Show $\langle p,q\rangle$ is an inner product.

i. $\langle q,p\rangle = \int_0^\infty q(x)p(x)e^{-x}\,dx = \int_0^\infty p(x)q(x)e^{-x}\,dx = \langle p,q\rangle$

ii. $\langle p,q+r\rangle = \int_0^\infty p(x)(q+r)(x)e^{-x}\,dx = \int_0^\infty p(x)q(x)e^{-x}\,dx + \int_0^\infty p(x)r(x)e^{-x}\,dx = \langle p,q\rangle + \langle p,r\rangle$

iii. $\langle p,aq\rangle = \int_0^\infty p(x))(aq)(x)e^{-x}\,dx = a\int_0^\infty p(x)q(x)e^{-x}\,dx = a\langle p,q\rangle$

iv. $\langle p,p\rangle = \int_0^\infty p(x)^2e^{-x}\,dx \geq 0$ because $p(x)^2e^{-x}$ is non-negative. Further

$$\langle p,p\rangle = 0 \quad \Rightarrow \quad \int_0^\infty p(x)^2e^{-x}\,dx = 0$$

$\Rightarrow \quad p(x)^2e^{-x} = 0$ because $p(x)^2e^{-x}$ is non-negative and continuous

$\Rightarrow \quad p(x) = 0$

So $\langle p,q\rangle$ is an inner product.

b. (15) Find the angle θ between the polynomials $p(x) = 1$ and $q(x) = x$.

You may use these integrals without proof:

$$\int_0^\infty e^{-x}\,dx = 1, \quad \int_0^\infty xe^{-x}\,dx = 1, \quad \int_0^\infty x^2e^{-x}\,dx = 2, \quad \int_0^\infty x^3e^{-x}\,dx = 6$$

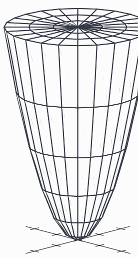
$\langle p,q\rangle = \int_0^\infty 1\cdot x\cdot e^{-x}\,dx = 1 \qquad \langle p,p\rangle = \int_0^\infty 1\cdot 1\cdot e^{-x}\,dx = 1 \qquad \langle q,q\rangle = \int_0^\infty x\cdot x\cdot e^{-x}\,dx = 2$

$$\cos\theta = \frac{\langle p,q\rangle}{|p||q|} = \frac{1}{1\cdot\sqrt{2}} = \frac{1}{\sqrt{2}} \quad \Rightarrow \quad \theta = 45° = \frac{\pi}{4}$$

4. (30 points) Gauss' Theorem states that if V is a volume in $\mathbf{R}^3$ and ∂V is its boundary surface oriented outward from V and $\vec{F}$ is a nice vector field on V then

$$\iiint_V \vec{\nabla} \cdot \vec{F}\, dV = \iint_{\partial V} \vec{F} \cdot \vec{dS}$$

Verify Gauss' Theorem if $\vec{F} = (xz^2, -yz^2, x^2z + y^2z)$ and V is the volume above the paraboloid $z = x^2 + y^2$ and below the plane $z = 9$. Notice that ∂V consists of the paraboloid P and a disk D. Be sure to use the correct orientations for P and D.



a. (6) Compute $\iiint_V \vec{\nabla} \cdot \vec{F}\, dV$:

i. $\vec{\nabla} \cdot \vec{F} = \qquad = \vec{\nabla} \cdot (xz^2, -yz^2, x^2z + y^2z) = z^2 - z^2 + x^2 + y^2 = x^2 + y^2$

$\vec{\nabla} \cdot \vec{F} = r^2$ (in cylindrical coordinates)

ii. $\iiint_V \vec{\nabla} \cdot \vec{F}\, dV = \qquad = \int_0^{2\pi} \int_0^3 \int_{r^2}^9 (r^2)\, r\, dz\, dr\, d\theta = 2\pi \int_0^3 \left[r^3 z \right]_{r^2}^9 dr$

$= 2\pi \int_0^3 (9r^3 - r^5)\, dr = 2\pi \left[\dfrac{9r^4}{4} - \dfrac{r^6}{6} \right]_0^3 = 2\pi \left(\dfrac{9 \cdot 3^4}{4} - \dfrac{3^6}{6} \right)$

$= 3^6 \pi \left(\dfrac{1}{2} - \dfrac{1}{3} \right) = \dfrac{3^6 \pi}{6} = \dfrac{243\pi}{2}$

b. (10) For the paraboloid P compute $\iint_P \vec{F} \cdot \vec{dS}$:

i. $\vec{R}(r,\theta) = \qquad = (r\cos\theta, r\sin\theta, r^2)$

ii. $\vec{R}_r = \qquad = (\cos\theta, \sin\theta, 2r)$

iii. $\vec{R}_\theta = \qquad = (-r\sin\theta, r\cos\theta, 0)$

iv. $\vec{N} = \qquad = \hat{\imath}(-2r^2\cos\theta) - \hat{\jmath}(2r^2\sin\theta) + \hat{k}(r\cos^2\theta + r\sin^2\theta) = (-2r^2\cos\theta, -2r^2\sin\theta, r)$

This points up. We need it down. Reverse it.

$\vec{N} = (2r^2\cos\theta, 2r^2\sin\theta, -r)$

v. $\vec{F}\big(\vec{R}(r,\theta)\big) = \qquad = (r^5\cos\theta, -r^5\sin\theta, r^4)$

vi. $\vec{F}\big(\vec{R}(r,\theta)\big) \cdot \vec{N} = \qquad = 2r^7\cos^2\theta - 2r^7\sin^2\theta - r^5$

vii. $\iint_P \vec{F} \cdot \vec{dS} = \qquad = \int_0^{2\pi} \int_0^3 (2r^7\cos^2\theta - 2r^7\sin^2\theta - r^5)\, dr\, d\theta = \int_0^3 \int_0^{2\pi} (2r^7\cos 2\theta - r^5)\, d\theta\, dr$

$= \int_0^3 \left[2r^7 \dfrac{\sin 2\theta}{2} - r^5\theta \right]_0^{2\pi} dr = -2\pi \int_0^3 r^5\, dr = -2\pi \dfrac{r^6}{6} \Big|_0^3 = -3^5\pi = -243\pi$

Problem **4** continued: Recall $\vec{F} = (xz^2, -yz^2, x^2z + y^2z)$.

c. (10) For the disk D compute $\iint_D \vec{F} \cdot \overrightarrow{dS}$:

i. $\vec{R}(r,\theta) = \qquad = (r\cos\theta, r\sin\theta, 9)$

ii. $\vec{R}_r = \qquad = \left(\ \cos\theta, \quad \sin\theta, \ 0 \right)$

iii. $\vec{R}_\theta = \qquad = \left(\left(-r\sin\theta, r\cos\theta, \ 0 \right) \right)$

iv. $\vec{N} = \qquad = (0, 0, r)$ This points up which is correct.

v. $\vec{F}\left(\vec{R}(r,\theta)\right) = \qquad = (81r\cos\theta, 81r\sin\theta, 9r^2)$

vi. $\vec{F}\left(\vec{R}(r,\theta)\right) \cdot \vec{N} = \qquad = 9r^3$

vii. $\iint_D \vec{F} \cdot \overrightarrow{dS} = \qquad = \int_0^{2\pi} \int_0^3 (9r^3)\, dr\, d\theta = 2\pi \frac{9r^4}{4} \Big|_0^3 = \frac{3^6\pi}{2} = \frac{729\pi}{2}$

d. (4) Verify the two sides of Gauss' Theorem are equal.

i. $\iiint_V \vec{\nabla} \cdot \vec{F}\, dV = \frac{243\pi}{2}$

ii. $\iint_P \vec{F} \cdot \overrightarrow{dS} + \iint_D \vec{F} \cdot \overrightarrow{dS} = -243\pi + \frac{729\pi}{2} = \frac{729\pi - 486\pi}{2} = \frac{243\pi}{2}$

iii. They are equal.