Name________________________________

Math 311 Final Spring 2010
Section 502 P. Yasskin

1	/26	4	/26
2	/26	5	/16
3	/12	Total	/106

1. (26 points) Let P_3 be the vector space of polynomials of degree less than 3.
Consider the linear operator $L : P_3 \to P_3$ given by $L(p) = \frac{1}{x}\int_0^x p(x)\,dx$.
In other words, $L(a+bx+cx^2) = \frac{1}{x}\left[ax + b\frac{x^2}{2} + c\frac{x^3}{3}\right]_0^x = a + b\frac{x}{2} + c\frac{x^2}{3}$.

a. (14 pts) Identify the domain, codomain, kernel and image, and the dimension of each.
Is L one-to-one? Why? Is L onto? Why?

b. (6 pts) Find the matrix of L relative to the basis $e_1 = 1$ $e_2 = x$ $e_3 = x^2$. Call it A.

c. (6 pts) Find the eigenvalues and eigenvectors of A. Find the eigenvalues and eigenpolynomials of L. No new computations!

2. (26 points) On the vector space P_3 consider the function of two polynomials given by

$$\langle p,q\rangle = p(-1)\ q(-1) + p(0)\ q(0) + p(1)\ q(1)$$

a. (10 pts) Show $\langle p,q\rangle$ is an inner product.

b. (16 pts) Apply the Gram-Schmidt procedure to the basis

$$e_1 = 1 \qquad e_2 = x \qquad e_3 = x^2$$

to produce an orthogonal basis w_1, w_2, w_3 and an orthonormal basis u_1, u_2, u_3.

HINT: If $p(x) = 1$, what are $p(-1)$, $p(0)$ and $p(1)$? What is $\langle 1,1\rangle$?

3. (12 pts) Let $y(x,t)$ denote the transverse displacement of an 8 cm string at position x and time t. The velocity of a wave on this string is measured as 3 cm/sec.

It is initially pulled to have the shape $f(x) = \begin{cases} 0.1(4+x) & \text{for} \quad -4 \le x \le 0 \\ 0.1(4-x) & \text{for} \quad 0 \le x \le 4 \end{cases}$

It is then released from rest at time $t = 0$. It is held fixed at both ends.

Write down the differential equation, boundary and initial conditions satisfied by the string.

Do not solve anything.

4. (26 pts) The heat equation for the temperature $z(x,t)$ on a 100 cm metal bar is

$$\frac{\partial z}{\partial t} = 9\frac{\partial^2 z}{\partial x^2}.$$

The temperature at the ends are held fixed at $25°C$ and $75°C$. Thus

$$z(0,t) = 25 \quad \text{and} \quad z(100,t) = 75 \qquad \forall t \geq 0$$

Initially, the temperature on the bar is

$$z(x,0) = 25 + \frac{x}{2} + 4\sin\left(\frac{7\pi x}{100}\right) \qquad \forall x \in [0,100]$$

Find the temperature $z(x,t)$ for $t \geq 0$ and $x \in [0,100]$.

HINT: First let $z(x,t) = 25 + \frac{x}{2} + y(x,t)$.

Write down the differential equation, boundary and initial conditions satisfied by $y(x,t)$.

Solve for $y(x,t)$ by separating variables. Then substitute back to get $z(x,t)$.

5. (16 pts) Find the fourier series for $f(x) = \begin{cases} 2+x & \text{for } -4 \le x \le 0 \\ 2-x & \text{for } 0 \le x \le 4 \end{cases}$

Then plot the function $f(x)$ and the first term of its fourier series.

HINT: The fourier series for $f(x)$ on the interval $[-L, L]$ is

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

where

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx \qquad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$