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Math 311 Final Spring 2010
Section 502 Solutions P. Yasskin

1	/26	4	/26
2	/26	5	/16
3	/12	Total	/106

1. (26 points) Let P_3 be the vector space of polynomials of degree less than 3.
Consider the linear operator $L : P_3 \to P_3$ given by $L(p) = \frac{1}{x}\int_0^x p(x)\,dx$.
In other words, $L(a+bx+cx^2) = \frac{1}{x}\left[ax + b\frac{x^2}{2} + c\frac{x^3}{3}\right]_0^x = a + b\frac{x}{2} + c\frac{x^2}{3}$.

a. (14 pts) Identify the domain, codomain, kernel and image, and the dimension of each. Is L one-to-one? Why? Is L onto? Why?

$Dom(L) = P_3 \qquad \dim Dom(L) = 3 \qquad Codom(L) = P_3 \qquad \dim Codom(L) = 3$

Kernel: If $p = a + bx + cx^2$ and $L(p) = 0$ then $a + b\frac{x}{2} + c\frac{x^2}{3} = 0$
or $a = b = c = 0$ or $p = 0$. $\quad Ker(L) = \{0\} \qquad \dim Ker(L) = 0$

Image: $Im(L) = \left\{a + b\frac{x}{2} + c\frac{x^2}{3}\right\} = Span(1, x, x^2) = P_3 \qquad \dim Im(L) = 3$

L is one-to-one because $Ker(L) = \{0\}$.

L is onto because $Im(L) = Codom(L) = P_3$.

b. (6 pts) Find the matrix of L relative to the basis $e_1 = 1 \quad e_2 = x \quad e_3 = x^2$. Call it A.

$$L(1) = 1 \qquad L(x) = \frac{x}{2} \qquad L(x^2) = \frac{x^2}{3}$$

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}$$

c. (6 pts) Find the eigenvalues and eigenvectors of A. Find the eigenvalues and eigenpolynomials of L. No new computations!

$$\lambda_1 = 1 \qquad \vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad p_1 = 1$$

$$\lambda_2 = \frac{1}{2} \qquad \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad p_2 = x$$

$$\lambda_3 = \frac{1}{3} \qquad \vec{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad p_3 = x^2$$

2. (26 points) On the vector space P_3 consider the function of two polynomials given by

$$\langle p,q\rangle = p(-1)\ q(-1) + p(0)\ q(0) + p(1)\ q(1)$$

a. (10 pts) Show $\langle p,q\rangle$ is an inner product.

i. $\langle q,p\rangle = q(-1)p(-1) + q(0)p(0) + q(1)p(1) = \langle p,q\rangle$

ii. $\langle ap+bq,r\rangle = [ap(-1)+bq(-1)]r(-1) + [ap(0)+bq(0)]r(0) + [ap(1)+bq(1)]r(1)$

$= a[p(-1)r(-1) + p(0)r(0) + p(1)r(1)] + b[q(-1)r(-1) + q(0)r(0) + q(1)r(1)]$

$= a\langle p,r\rangle + b\langle q,r\rangle$

iii. $\langle p,p\rangle = p(-1)^2 + p(0)^2 + p(1)^2 \geq 0$ and $= 0$ only if $p(-1) = p(0) = p(1) = 0$

If $p = a + bx + cx^2$, then $p(-1) = a - b + c = 0$ $\quad p(0) = a = 0$ $\quad p(1) = a + b + c = 0$

So $a = 0,$ $\ -b + c = 0,$ $\ b + c = 0$ which says $a = b = c = 0$ or $p = 0$.

b. (16 pts) Apply the Gram-Schmidt procedure to the basis

$$e_1 = 1 \qquad e_2 = x \qquad e_3 = x^2$$

to produce an orthogonal basis w_1, w_2, w_3 and an orthonormal basis u_1, u_2, u_3.

HINT: If $p(x) = 1$, what are $p(-1)$, $p(0)$ and $p(1)$? What is $\langle 1,1\rangle$?

$w_1 = e_1 = \boxed{1}$

$\langle w_1,w_1\rangle = \langle 1,1\rangle = 1\cdot 1 + 1\cdot 1 + 1\cdot 1 = 3 \qquad |w_1| = \sqrt{3}$

$\langle e_2,w_1\rangle = \langle x,1\rangle = (-1)\cdot 1 + 0\cdot 1 + 1\cdot 1 = 0$

$w_2 = e_2 - \dfrac{\langle e_2,w_1\rangle}{\langle w_1,w_1\rangle} w_1 = \boxed{x}$

$\langle w_2,w_2\rangle = \langle x,x\rangle = (-1)\cdot(-1) + 0\cdot 0 + 1\cdot 1 = 2 \qquad |w_2| = \sqrt{2}$

$\langle e_3,w_1\rangle = \langle x^2,1\rangle = 1\cdot 1 + 0\cdot 1 + 1\cdot 1 = 2$

$\langle e_3,w_2\rangle = \langle x^2,x\rangle = 1\cdot(-1) + 0\cdot 0 + 1\cdot 1 = 0$

$w_3 = e_3 - \dfrac{\langle e_3,w_1\rangle}{\langle w_1,w_1\rangle} w_1 - \dfrac{\langle e_3,w_2\rangle}{\langle w_2,w_2\rangle} w_2 = x^2 - \dfrac{2}{3}1 - 0 = \boxed{x^2 - \dfrac{2}{3}}$

$\langle w_3,w_3\rangle = \left((-1)^2 - \dfrac{2}{3}\right)^2 + \left(-\dfrac{2}{3}\right)^2 + \left(1^2 - \dfrac{2}{3}\right)^2 = \dfrac{1}{9} + \dfrac{4}{9} + \dfrac{1}{9} = \dfrac{2}{3} \qquad |w_3| = \sqrt{\dfrac{2}{3}}$

$u_1 = \dfrac{w_1}{|w_1|} = \boxed{\dfrac{1}{\sqrt{3}}} \qquad u_2 = \dfrac{w_2}{|w_2|} = \boxed{\dfrac{x}{\sqrt{2}}} \qquad u_3 = \dfrac{w_3}{|w_3|} = \boxed{\sqrt{\dfrac{3}{2}}\left(x^2 - \dfrac{2}{3}\right)}$

3. (12 pts) Let $y(x,t)$ denote the transverse displacement of an 8 cm string at position x and time t. The velocity of a wave on this string is measured as 3 cm/sec.

It is initially pulled to have the shape $f(x) = \begin{cases} 0.1(4+x) & \text{for } -4 \le x \le 0 \\ 0.1(4-x) & \text{for } 0 \le x \le 4 \end{cases}$

It is then released from rest at time $t = 0$. It is held fixed at both ends.

Write down the differential equation, boundary and initial conditions satisfied by the string.

Do not solve anything.

The wave equation with velocity 3 is $\dfrac{\partial^2 y}{\partial t^2} = 9\dfrac{\partial^2 y}{\partial x^2}$.

The boundary conditions are $y(-4,t) = 0$ and $y(4,t) = 0$ $\forall t \ge 0$.

The initial conditions are $y(x,0) = f(x)$ and $\dfrac{\partial y}{\partial t}(x,0) = 0$ $\forall x \in [-4,4]$.

4. (26 pts) The heat equation for the temperature $z(x,t)$ on a 100 cm metal bar is

$$\frac{\partial z}{\partial t} = 9\frac{\partial^2 z}{\partial x^2}.$$

The temperature at the ends are held fixed at $25°C$ and $75°C$. Thus

$$z(0,t) = 25 \quad \text{and} \quad z(100,t) = 75 \quad \forall t \ge 0$$

Initially, the temperature on the bar is

$$z(x,0) = 25 + \frac{x}{2} + 4\sin\left(\frac{7\pi x}{100}\right) \quad \forall x \in [0,100]$$

Find the temperature $z(x,t)$ for $t \ge 0$ and $x \in [0,100]$.

HINT: First let $z(x,t) = 25 + \frac{x}{2} + y(x,t)$.

Write down the differential equation, boundary and initial conditions satisfied by $y(x,t)$.

Solve for $y(x,t)$ by separating variables. Then substitute back to get $z(x,t)$.

Let $z(x,t) = 25 + \frac{x}{2} + y(x,t)$. Then

$\frac{\partial z}{\partial t} = \frac{\partial y}{\partial t}$ $\quad \frac{\partial z}{\partial x} = \frac{1}{2} + \frac{\partial y}{\partial x}$ $\quad \frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 y}{\partial x^2}$ So the differential equation is

$$\frac{\partial y}{\partial t} = 9\frac{\partial^2 y}{\partial x^2}.$$

$z(0,t) = 25 + y(0,t)$ $\quad z(100,t) = 75 + y(100,t)$ So the boundary conditions are

$$y(0,t) = 0 \quad \text{and} \quad y(100,t) = 0.$$

$z(x,0) = 25 + \frac{x}{2} + y(x,0)$ So the initial condition is

$$y(x,0) = 4\sin\left(\frac{7\pi x}{100}\right)$$

To separate variables, let $y(x,t) = X(x)T(t)$. Substitute into the differential equation and divide by XT:

$$\frac{1}{9T}\frac{dT}{dt} = \frac{1}{X}\frac{d^2X}{dx^2}.$$

Since the left is a function of t and the right is a function of x, they both must equal a constant.

This constant must be negative so that T does not grow exponentially. So

$$\frac{1}{9T}\frac{dT}{dt} = \frac{1}{X}\frac{d^2X}{dx^2} = -\lambda^2$$

or

$$\frac{dT}{dt} = -9\lambda^2 T \quad \text{and} \quad \frac{d^2X}{dx^2} = -\lambda^2 X$$

The solutions are

$$T = Ae^{-9\lambda^2 t} \quad \text{and} \quad X = P\sin(\lambda x) + Q\cos(\lambda x)$$

We first satisfy the boundary conditions.

$y(0,t) = 0$ implies $X(0) = Q = 0$ or $X = P\sin(\lambda x)$

$y(100,t) = 0$ implies $X(100) = P\sin(100\lambda) = 0$. So $\lambda = \frac{n\pi}{100} \equiv \lambda_n$.

By superposition, a solution of the differential equation satisfying the boundary conditions is

$$y(x,t) = \sum_{n=1}^{\infty} P_n \sin(\lambda_n x)e^{-9\lambda_n^2 t} = \sum_{n=1}^{\infty} P_n \sin\left(\frac{n\pi x}{100}\right)\exp\left(-9\left(\frac{n\pi}{100}\right)^2 t\right)$$

The initial condition says

$$y(x,0) = \sum_{n=1}^{\infty} P_n \sin\left(\frac{n\pi x}{100}\right) = 4\sin\left(\frac{7\pi x}{100}\right)$$

Comparing, we see $P_7 = 4$ and all other P_n's are 0. So the solution is

$$y(x,t) = 4\sin(\lambda_7 x)e^{-9\lambda_7^2 t} = 4\sin\left(\frac{7\pi x}{100}\right)\exp\left(-9\left(\frac{7\pi}{100}\right)^2 t\right).$$

Substitute back to get

$$z(x,t) = 25 + \frac{x}{2} + 4\sin(\lambda_7 x)e^{-9\lambda_7^2 t} = 25 + \frac{x}{2} + 4\sin\left(\frac{7\pi x}{100}\right)\exp\left(-9\left(\frac{7\pi}{100}\right)^2 t\right).$$

5. (16 pts) Find the fourier series for $f(x) = \begin{cases} 2+x & \text{for } -4 \le x \le 0 \\ 2-x & \text{for } 0 \le x \le 4 \end{cases}$

Then plot the function $f(x)$ and the first term of its fourier series.

HINT: The fourier series for $f(x)$ on the interval $[-L, L]$ is

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

where

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx \qquad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

The interval is $[-4, 4]$. So $L = 4$.

The function $f(x)$ is even because for $a \ge 0$, $f(a) = 2-a$ while $f(-a) = 2+(-a) = 2-a = f(a)$. So only the $\cos$ terms are non-zero.

$$a_0 = \frac{1}{4}\int_{-4}^{4} f(x)\cos(0)\,dx = \frac{1}{4}\int_{-4}^{0}(2+x)\,dx + \frac{1}{4}\int_{0}^{4}(2-x)\,dx = \frac{1}{2}\int_{0}^{4}(2-x)\,dx = \frac{1}{2}\left[-\frac{(2-x)^2}{2}\right]_0^4$$

$$= \frac{1}{2}\left[-\frac{(-2)^2}{2}\right] - \frac{1}{2}\left[-\frac{(2)^2}{2}\right] = 0$$

$$a_n = \frac{1}{4}\int_{-4}^{4} f(x)\cos\left(\frac{n\pi x}{4}\right)dx = \frac{1}{4}\int_{-4}^{0}(2+x)\cos\left(\frac{n\pi x}{4}\right)dx + \frac{1}{4}\int_{0}^{4}(2-x)\cos\left(\frac{n\pi x}{4}\right)dx$$

$$= \frac{1}{2}\int_{0}^{4}(2-x)\cos\left(\frac{n\pi x}{4}\right)dx$$ use integration by parts: $u = 2-x \quad dv = \cos\left(\frac{n\pi x}{4}\right)dx$, $du = -dx \quad v = \frac{4}{n\pi}\sin\left(\frac{n\pi x}{4}\right)$

$$a_n = \frac{1}{2}\left[(2-x)\frac{4}{n\pi}\sin\left(\frac{n\pi x}{4}\right) + \frac{4}{n\pi}\int \sin\left(\frac{n\pi x}{4}\right)dx\right]_0^4 = \frac{1}{2}\left[-\left(\frac{4}{n\pi}\right)^2\cos\left(\frac{n\pi x}{4}\right)\right]_0^4$$

$$= -\frac{1}{2}\left(\frac{4}{n\pi}\right)^2[\cos(n\pi) - \cos(0)] = -\frac{1}{2}\left(\frac{4}{n\pi}\right)^2 \cdot \begin{cases} 0 & \text{for } n \text{ even} \\ -2 & \text{for } n \text{ odd} \end{cases} = \begin{cases} 0 & \text{for } n \text{ even} \\ \frac{16}{n^2\pi^2} & \text{for } n \text{ odd} \end{cases}$$

$$f(x) \approx \sum_{\substack{n=1 \\ odd}}^{\infty} \frac{16}{n^2\pi^2}\cos\left(\frac{n\pi x}{4}\right)$$

$$= \frac{16}{\pi^2}\cos\left(\frac{\pi x}{4}\right) + \frac{16}{9\pi^2}\cos\left(\frac{3\pi x}{4}\right) + \cdots$$