MATH 311 Section 502 Homework on Inner Products Spring 2010 P. Yasskin

Consider the vector space
$F = \text{Span}(1, \sin x, \cos x, \sin 2x, \cos 2x, \sin 3x, \cos 3x, \cdots, \sin px, \cos px, \cdots)$
where p is an integer, with the usual addition and scalar multiplication of functions and the subspace $F_2 = \text{Span}(1, \sin x, \cos x, \sin 2x, \cos 2x)$. An inner product for F or F_2 is

$$\langle f, g \rangle = \int_0^{2\pi} f(x) g(x)\, dx$$

1. Compute each of the following inner products:

$$\langle 1, 1 \rangle = \int_0^{2\pi} 1\, dx$$

$$\langle 1, \sin px \rangle = \int_0^{2\pi} \sin px\, dx \qquad \langle 1, \cos px \rangle = \int_0^{2\pi} \cos px\, dx$$

$$\langle \sin px, \sin px \rangle = \int_0^{2\pi} \sin^2 px\, dx \qquad \langle \cos px, \cos px \rangle = \int_0^{2\pi} \cos^2 px\, dx$$

$$\langle \sin px, \cos px \rangle = \int_0^{2\pi} \sin px \cos px\, dx$$

$$\langle \sin px, \sin qx \rangle = \int_0^{2\pi} \sin px \sin qx\, dx \qquad \langle \cos px, \cos qx \rangle = \int_0^{2\pi} \cos px \cos qx\, dx$$

$$\langle \sin px, \cos qx \rangle = \int_0^{2\pi} \sin px \cos qx\, dx$$

Assume p and q are positive integers and $p \neq q$.
Be sure to show any identities you use and the antiderivatives.
Do not just use a computer, calculator or table of integrals.

2. Are the vectors $1, \sin x, \cos x, \sin 2x, \cos 2x, \sin 3x, \cos 3x, \cdots, \sin px, \cos px, \cdots$ orthonormal, orthogonal or neither. Why?

3. Consider the functions

$$f = \sin x + 2\sin^2 x \quad \text{and} \quad g = \cos x + 2\cos^2 x.$$

Compute $\langle f, g \rangle$ by directly computing the integral.

4. Find the matrix of the inner product $\underset{e \leftrightarrow e}{G}$ on F_2 relative to the basis $e = (1, \sin x, \cos x, \sin 2x, \cos 2x)$.
In other words, find the 5×5 matrix $\underset{e \leftrightarrow e}{G} = (g_{ij})$ whose entries are $g_{ij} = \langle e_i, e_j \rangle$.

5. Find the components of f and g (from #3) relative to the basis $e = (1, \sin x, \cos x, \sin 2x, \cos 2x)$. Recall these are called $(f)_e$ and $(g)_e$.
HINT: What are the identities for $\sin^2 x$ and $\cos^2 x$?

6. Recompute $\langle f, g \rangle$ but use the matrix of the inner product and the components of f and g.
HINT: $\langle f, g \rangle = (f)_e^T \underset{e \leftrightarrow e}{G} (g)_e$