Consider the vector space of 2×2 matrices $M(2,2)$ with standard basis

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad E_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad E_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

and the linear operator

$$L : M(2,2) \to M(2,2) : L(X) = PX \qquad \text{where} \qquad P = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$

NOTE: P is NOT the matrix of this linear map!

1. What is $\dim M(2,2)$? What is the size of the matrix A of the linear map L? Find the matrix of the linear map L relative to the E basis. Call it $\underset{E\leftarrow E}{A}$.

2. Show the eigenvalues of $\underset{E\leftarrow E}{A}$ are $\lambda = 2$ and $\lambda = 4$.
HINT: Use long division to factor out $(\lambda - 2)$ and $(\lambda - 4)$.

3. Find the eigenvectors for the eigenvalue $\lambda = 2$.
Define $\vec{v}_1$ and $\vec{v}_2$ to be a basis for the eigenspace.

4. Find the eigenvector(s) for the eigenvalue $\lambda = 4$.
Define $\vec{v}_3$ and $\vec{v}_4$ to be a basis for the eigenspace.

5. Each of the eigenvectors $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ are the component vectors of eigenmatrices V_1, V_2, V_3, V_4 relative the $E = (E_1, E_2, E_3, E_4)$ basis. Hook the components onto the basis vectors to produce the eigenmatrices V_1, V_2, V_3, V_4. Verify they are eigenmatrices by checking that $L(V_k) = \lambda V_k$ using the definition $L(X) = PX$.

6. The matrices $V = (V_1, V_2, V_3, V_4)$ form a second basis for $M(2,2)$.
Find the matrix of the linear map L relative to the V basis. Call it $\underset{V \leftarrow V}{D}$.

7. Find the change of basis matrices $\underset{E \leftarrow V}{C}$ and $\underset{V \leftarrow E}{C}$.

8. Verify your matrices satisfy $\underset{E \leftarrow V}{C} \underset{V \leftarrow V}{D} \underset{V \leftarrow E}{C} = \underset{E \leftarrow E}{A}$.

9. Compute $\left(\underset{E \leftarrow E}{A}\right)^4$