Name________________________________

Math 311 Exam 1 Spring 2013

Section 501 Solutions P. Yasskin

1	/25	4	/10
2	/20	5	/15
3	/30	Total	/100

1. (25 points) Let $A = \begin{pmatrix} 1 & 3 & 0 & 2 & 2 \\ 2 & 7 & 2 & 5 & 5 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 3 & 6 & 4 & 3 \end{pmatrix}$, $\vec{x} = \begin{pmatrix} p \\ q \\ x \\ y \\ z \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 2 \\ 6 \\ 3 \\ 8 \end{pmatrix}$, $\vec{c} = \begin{pmatrix} 2 \\ 6 \\ 3 \\ 7 \end{pmatrix}$.

Solve both equations $A\vec{x} = \vec{b}$ and $A\vec{x} = \vec{c}$. Give all solutions or say why there are no solutions.

SOLUTION: The first column shows the row reduction of the augmented matrix with both right hand sides. The second column discusses the solutions.

$$\left(\begin{array}{ccccc|cc} 1 & 3 & 0 & 2 & 2 & 2 & 2 \\ 2 & 7 & 2 & 5 & 5 & 6 & 6 \\ 0 & 1 & 2 & 2 & 1 & 3 & 3 \\ 0 & 3 & 6 & 4 & 3 & 8 & 7 \end{array}\right) \begin{array}{l} \\ R2 - 2 \cdot R1 \\ \\ \\ \end{array}$$

$A\vec{x} = \vec{b}$ has no solution because the last equation is $0 = 1$.

$$\left(\begin{array}{ccccc|cc} 1 & 3 & 0 & 2 & 2 & 2 & 2 \\ 0 & 1 & 2 & 1 & 1 & 2 & 2 \\ 0 & 1 & 2 & 2 & 1 & 3 & 3 \\ 0 & 3 & 6 & 4 & 3 & 8 & 7 \end{array}\right) \begin{array}{l} R1 - 3 \cdot R2 \\ \\ R3 - R2 \\ R4 - 3 \cdot R2 \end{array}$$

$A\vec{x} = \vec{c}$ has free variables x and z.

$$\left(\begin{array}{ccccc|cc} 1 & 0 & -6 & -1 & -1 & -4 & -4 \\ 0 & 1 & 2 & 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 2 & 1 \end{array}\right) \begin{array}{l} R1 + R3 \\ R2 - R3 \\ \\ R4 - R3 \end{array}$$

$A\vec{x} = \vec{c}$ equivalent equations:

$p - 6x - z = -3$

$q + 2x + z = 1$

$y = 1$

$$\left(\begin{array}{ccccc|cc} 1 & 0 & -6 & 0 & -1 & -3 & -3 \\ 0 & 1 & 2 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

$A\vec{x} = \vec{c}$ solution:

$p = -3 + 6r + s$

$q = 1 - 2r - s$

$x = r$

$y = 1$

$z = s$

2. (20 points) Let $A = \begin{pmatrix} 1 & 3 & 0 & 2 & 2 \\ 2 & 7 & 2 & 5 & 5 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 3 & 6 & 4 & 3 \end{pmatrix}$ as in problem 1.

a. Find a basis for $N(A)$, the null space of A. What is $\dim N(A)$, the nullity of A?

SOLUTION: The augmented matrix and the reduced matrix are:

$$\left(\begin{array}{ccccc|c} 1 & 3 & 0 & 2 & 2 & 0 \\ 2 & 7 & 2 & 5 & 5 & 0 \\ 0 & 1 & 2 & 2 & 1 & 0 \\ 0 & 3 & 6 & 4 & 3 & 0 \end{array}\right) \Rightarrow \left(\begin{array}{ccccc|c} 1 & 0 & -6 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

The solution is $\begin{pmatrix} p \\ q \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6r+s \\ -2r-s \\ r \\ 0 \\ s \end{pmatrix} = r\begin{pmatrix} 6 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

So $(6,-2,1,0,0)^{\mathsf{T}}$ and $(1,-1,0,0,1)^{\mathsf{T}}$ are a basis for $N(A)$, and $\dim N(A) = 2$.

b. Find a basis for $R(A)$, the row space of A. What is $\dim R(A)$, the row rank of A?

SOLUTION: $R(A) = Span(\text{rows of } A)$ Row operations simply change the spanning vectors.
So $R(A) = Span((1,0,-6,0,1),(0,1,2,0,1),(0,0,0,1,0))$
So $(1,0,-6,0,1)$, $(0,1,2,0,1)$ and $(0,0,0,1,0)$ are a basis for $R(A)$, and $\dim R(A) = 3$.

c. Find a basis for $C(A)$, the column space of A. What is $\dim C(A)$, the column rank of A?

SOLUTION: The columns with leading 1's in the row reduction are the columns in A which are basis vectors.
So $(1,2,0,0)^{\mathsf{T}}$, $(3,7,1,3)^{\mathsf{T}}$ and $(2,5,2,4)^{\mathsf{T}}$ are a basis for $C(A)$, and $\dim C(A) = 3$.

d. Give 2 relations between the 3 numbers $\dim N(A)$, $\dim R(A)$ and $\dim C(A)$ which would be true for any 4×5 matrix A. (No proof.)

SOLUTION: $\dim N(A) + \dim R(A) = 5$ $\qquad \dim R(A) = \dim C(A)$

3. (30 points) Let $A \in M(n,n)$ and $\vec{b} \in \mathbb{R}^n$. For each of the following conditions, say whether A is singular or non-singular (circle one). Then give a reason. For one and only one of these, you may say this is the definition of singular or non-singular. For the rest, your reason must say why the given condition is equivalent to the definition.

a. A is row reducable to the unit matrix. singular **[non-singular]**

Reason: The inverse is found by row reducing $(A|\mathbf{1})$ to $(\mathbf{1}|A)$. So A is invertible if and only if it is row reducable to the unit matrix.

b. The reduced row echelon form of A has a row of zeros at the bottom. **[singular]** non-singular

Reason: The inverse is found by row reducing $(A|\mathbf{1})$ to $(\mathbf{1}|A)$. So A is non-invertible if and only if it's row reduction has a row of zeros at the bottom.

c. A is invertible. singular **[non-singular]**

Reason: This is the definition.

d. The equation $A\vec{x} = \vec{0}$ has an infinite number of solutions. **[singular]** non-singular

Reason: If A is non-invertible, then the row reduction of A has a row of zeros at the bottom. Since A is square, there must be a free variable. Since $\vec{x} = \vec{0}$ is a solution, there must be an infinite number of solutions.

e. The equation $A\vec{x} = \vec{b}$ has a unique solution. singular **[non-singular]**

Reason: If A is invertible, then the unique solution is $\vec{x} = A^{-1}\vec{b}$.

f. $\det(A) = 0$ **[singular]** non-singular

Reason: The $\det A$ may be computed by using row operations. If it is row reducible to the unit matrix, the determinant is non-zero (see d above). If it's row reduction has a row of zeros at the bottom, the determinant is zero (see e above).

4. $(10$ points$)$ Recall the definitions:

If A is a $p \times q$ matrix then A^{T} is the $q \times p$ matrix whose ij entry is $\left(A^{\mathsf{T}}\right)_{ij} = A_{ji}$.

If A is a $p \times q$ matrix and B is a $q \times r$ matrix then AB is the $p \times r$ matrix whose ij entry is $(AB)_{ij} = \sum_{k=0}^{q} A_{ik} B_{kj}$.

Use only these definitions to prove $(AB)^{\mathsf{T}} = B^{\mathsf{T}} A^{\mathsf{T}}$.

SOLUTION:

$$\left[(AB)^{\mathsf{T}}\right]_{ij} = (AB)_{ji} = \sum_{k=0}^{q} A_{jk} B_{ki} = \sum_{k=0}^{q} \left(A^{\mathsf{T}}\right)_{kj} \left(B^{\mathsf{T}}\right)_{ik} = \sum_{k=0}^{q} \left(B^{\mathsf{T}}\right)_{ik} \left(A^{\mathsf{T}}\right)_{kj} = \left(B^{\mathsf{T}} A^{\mathsf{T}}\right)_{ij}$$

5. $(15$ points$)$ In the vector space $V = \mathbb{R}^{+}$ with $a \oplus b = ab$ and $p \odot a = a^{p}$, write out each of the following facts as identities about ordinary multiplication and exponentiation.

a. $p \odot (a \oplus b) = (p \odot a) \oplus (p \odot b)$.. SOLUTION: $(ab)^{p} = a^{p} b^{p}$

b. $(p + q) \odot a = (p \odot a) \oplus (q \odot a)$.. SOLUTION: $a^{p+q} = a^{p} a^{q}$

c. $(pq) \odot a = p \odot (q \odot a)$... SOLUTION: $a^{pq} = (a^{q})^{p}$

d. $0 \otimes a = \vec{0}$.. SOLUTION: $a^{0} = 1$

e. $p \otimes \vec{0} = \vec{0}$.. SOLUTION: $1^{p} = 1$