1	/40
2	/30
3	/40
Total	/110

MATH 311 Section 501 Exam 2 Spring 2013 P. Yasskin

1. (35 points+5 e.c.) Consider the vector space $M(2,2)$ of 2×2 real matrices.
Consider the bases

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad E_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad E_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$F_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad F_2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \quad F_3 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \quad F_4 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

Consider the function $L : M(2,2) \to M(2,2)$ given by

$$L(X) = 2X - X^{\mathsf{T}}$$

a. (5 pts) Show L is linear.

Solution:

$$L(aX+bY) = 2(aX+bY) - (aX+bY)^{\mathsf{T}} = a\left[2(X) - (X)^{\mathsf{T}}\right] + b\left[2(Y) - (Y)^{\mathsf{T}}\right] = aL(X) + bL(Y)$$

b. (5 pts) Find the matrix of L relative to the E-basis. Call it $\underset{E\leftarrow E}{A}$.

HINT: A is NOT a 2×2 matrix!

Solution:

$$L(E_1) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = E_1$$

$$L(E_2) = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ -1 & 0 \end{pmatrix} = 2E_2 - E_3$$

$$L(E_3) = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix} = -E_2 + 2E_3$$

$$L(E_4) = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = E_4$$

$$\underset{E\leftarrow E}{A} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

c. (5 pts) Find the change of basis matrix $\underset{E\leftarrow F}{C}$ from the F-basis to the E-basis.

Solution:

$$F_1 = E_1$$
$$F_2 = E_1 + E_2$$
$$F_3 = E_1 + E_2 + E_3$$
$$F_4 = E_1 + E_2 + E_3 + E_4$$

$$\underset{E\leftarrow F}{C} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

d. (5 pts) Find the change of basis matrix $\underset{F\leftarrow E}{C}$ from the E-basis to the F-basis.

Solution:

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} R1-R4 \\ R2-R4 \\ R3-R4 \\ \end{array} \Rightarrow \left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 0 & 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} R1-R3 \\ R2-R3 \\ \\ \end{array} \Rightarrow$$

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} R1-R2 \\ \\ \\ \end{array} \Rightarrow \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\underset{F\leftarrow E}{C} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

e. (10 pts) Find the matrix of L relative to the F-basis. Call it $\underset{F\leftarrow F}{B}$.

Solution:

$$\underset{F\leftarrow F}{B} = \underset{F\leftarrow E}{C}\ \underset{E\leftarrow E}{A}\ \underset{E\leftarrow F}{C} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

f. (5 pts) Find $\underset{F\leftarrow F}{B}$ by a second method.

Solution:

$$L(F_1) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = F_1$$

$$L(F_2) = \begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix}$$

$$= aF_1 + bF_2 + cF_3 + dF_4 = \begin{pmatrix} a+b+c+d & b+c+d \\ c+d & d \end{pmatrix}$$

$d = 0,\quad c = -1,\quad b = 3,\quad a = -1 \qquad L(F_2) = -F_1 + 3F_2 - F_3$

$$L(F_3) = \begin{pmatrix} 2 & 2 \\ 2 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = F_3$$

$$L(F_4) = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = F_4$$

$$\underset{F\leftarrow F}{B} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

g. (5 pts Extra Credit) Looking at your solutions to (b), (e) and (f), find 3 linearly independent eigenvectors of L. What are their eigenvalues?

Solution:

$L(F_1) = F_1 \qquad L(F_3) = F_3 \qquad L(F_4) = F_4$

F_1, F_3, F_4 are linearly independent since they are part of a basis.

The eigenvalues are all 1.

2. (30 points) Consider the vector space $M(2,2)$ of 2×2 real matrices. Consider the function of two matrices

$$\langle X, Y\rangle = \mathrm{tr}\left(XDY^{\mathsf{T}}\right) \quad \text{where} \quad D = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

where tr means trace (sum of diagonal elements) and $^{\mathsf{T}}$ means transpose.

a. (10 pts) Show $\langle X, Y\rangle$ is an inner product.

HINT: First compute the inner product of $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $Y = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$.

Solution:

$$\langle X, Y\rangle = \mathrm{tr}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} e & g \\ f & h \end{pmatrix}\right) = \mathrm{tr}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 2e & 2g \\ f & h \end{pmatrix}\right)$$

$$= \mathrm{tr}\begin{pmatrix} 2ae+bf & 2ag+bh \\ 2ce+df & 2cg+dh \end{pmatrix} = 2ae+bf+2cg+dh$$

(1) $\langle X, X\rangle = 2a^2 + b^2 + 2c^2 + d^2 \geq 0$ and $= 0$ iff $a = b = c = d = 0$, i.e. $X = \mathbf{0}$

(2) $\langle Y, X\rangle = 2ea + fb + 2gc + hd = 2ae + bf + 2cg + dh = \langle X, Y\rangle$

(3) $\langle X, aY + bZ\rangle = \mathrm{tr}\left(XD(aY+bZ)^{\mathsf{T}}\right) = \mathrm{tr}\left(aXDY^{\mathsf{T}} + bXDZ^{\mathsf{T}}\right)$

$= a\,\mathrm{tr}\left(XDY^{\mathsf{T}}\right) + b\,\mathrm{tr}\left(XDZ^{\mathsf{T}}\right) = a\langle X, Y\rangle + b\langle X, Z\rangle$

b. (10 pts) Find the angle between the matrices $P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $Q = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Solution: $\langle P, Q\rangle = 2 \cdot 1 \cdot 1 + 0 \cdot 0 + 2 \cdot 0 \cdot 0 + 1 \cdot (-1) = 1$

$|P| = \sqrt{\langle P, P\rangle} = \sqrt{2 \cdot 1 \cdot 1 + 0 \cdot 0 + 2 \cdot 0 \cdot 0 + 1 \cdot 1} = \sqrt{3}$

$|Q| = \sqrt{\langle Q, Q\rangle} = \sqrt{2 \cdot 1 \cdot 1 + 0 \cdot 0 + 2 \cdot 0 \cdot 0 + (-1) \cdot (-1)} = \sqrt{3}$

$\cos\theta = \dfrac{\langle P, Q\rangle}{|P||Q|} = \dfrac{1}{3} \qquad \theta = \arccos\left(\dfrac{1}{3}\right)$

c. (10 pts) Let $V = \mathrm{Span}\{P, Q\}$ where P and Q are given in (b).

- Find $V^{\perp}$ the orthogonal subspace to V within $M(2,2)$.

Solution: $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is in $V^{\perp}$ iff $\langle X, P\rangle = 0$ and $\langle X, Q\rangle = 0$, or

$2a + d = 0$ and $2a - d = 0$, or $a = d = 0$, or $Y = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} = b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

So $V^{\perp} = \left\{\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}\right\} = Span\left\{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right\}$

- Find a basis for $V^{\perp}$ and its dimension.

Solution: A basis is $E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $E_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $\qquad \dim V^{\perp} = 2$

3. (35 points+5 e.c.) Consider the matrix $A = \begin{pmatrix} 0 & 6 \\ -1 & 5 \end{pmatrix}$.

a. (15 pts) Find the eigenvalues and corresponding eigenvectors of A.

Solution: $\det(A - \lambda \mathbf{1}) = \det\begin{pmatrix} -\lambda & 6 \\ -1 & 5-\lambda \end{pmatrix} = -\lambda(5-\lambda) + 6 = \lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3) = 0$

$\lambda = 2, 3$

For $\lambda = 2$: $\left(\begin{array}{cc|c} -2 & 6 & 0 \\ -1 & 3 & 0 \end{array}\right) \Rightarrow \left(\begin{array}{cc|c} 1 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3r \\ r \end{pmatrix} \Rightarrow \vec{v}_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

For $\lambda = 3$: $\left(\begin{array}{cc|c} -3 & 6 & 0 \\ -1 & 2 & 0 \end{array}\right) \Rightarrow \left(\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2r \\ r \end{pmatrix} \Rightarrow \vec{v}_3 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

b. (10 pts) Find the diagonalizing matrix X so that $A = XDX^{-1}$ where D is diagonal. What are X^{-1} and D? Just state them. No derivation.

Solution: $X = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$ $\quad X^{-1} = \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix}$ $\quad D = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$

c. (5 pts) Find A^4.

Solution: $A^4 = XD^4X^{-1} = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 16 & 0 \\ 0 & 81 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 16 & -32 \\ -81 & 243 \end{pmatrix}$

$= \begin{pmatrix} 48 - 162 & -96 + 486 \\ 16 - 81 & -32 + 243 \end{pmatrix} = \begin{pmatrix} -114 & 390 \\ -65 & 211 \end{pmatrix}$

d. (5 pts) Find $\cos(\pi A)$.

Solution: $\cos(\pi A) = X\cos(\pi D)X^{-1} = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \cos 2\pi & 0 \\ 0 & \cos 3\pi \end{pmatrix}\begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix}$

$= \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 1 & -3 \end{pmatrix} = \begin{pmatrix} 5 & -12 \\ 2 & -5 \end{pmatrix}$

e. (5 pts Extra Credit) What are the eigenvalues and corresponding eigenvectors of $5A$?

Solution: $A\begin{pmatrix} 3 \\ 1 \end{pmatrix} = 2\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $A\begin{pmatrix} 2 \\ 1 \end{pmatrix} = 3\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

So $5A\begin{pmatrix} 3 \\ 1 \end{pmatrix} = 10\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $5A\begin{pmatrix} 2 \\ 1 \end{pmatrix} = 15\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\lambda = 10$ and $\vec{v}_{10} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$. $\quad \lambda = 15$ and $\vec{v}_{15} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.