MATH 311 Section 501 Spring 2013 P. Yasskin

Final - Solutions

1. (20 points) Hams Duet is flying the Millenium Eagle through a region of intergalactic space containing a deadly hyperon vector field which is a function of position, $\vec{H} = (H_1(x,y,z), H_2(x,y,z), H_3(x,y,z))$. Of course, its magnitude is $M = \left|\vec{H}\right| = \sqrt{(H_1)^2 + (H_2)^2 + (H_3)^2}$. At stardate time $t = 21437.439$ years, Hams is located at the point $(x,y,z) = (5,-3,2)$ millilightyears and has velocity $\vec{v} = (3,-2,1)$ millilightyears/year. At that instant, he measures the hyperon density is

$$\vec{H} = (12\times 10^4, -3\times 10^4, 4\times 10^4) \quad \text{hyperons/millilightyear}^3$$

the gradients of its components are

$$\vec{\nabla} H_1 = (2,-1,3) \qquad \vec{\nabla} H_2 = (4,0,-1) \qquad \vec{\nabla} H_3 = (-2,1,3) \quad \text{hyperons/millilightyear}^4.$$

Find the **current hyperon magnitude** M and its **current rate of change** $\dfrac{dM}{dt}$ as seen by Hams?

HINT: Compute M and then the Jacobian matrices $\dfrac{D(M)}{D(H_1,H_2,H_3)}$, $\dfrac{D(H_1,H_2,H_3)}{D(x,y,z)}$ and $\dfrac{D(x,y,z)}{D(t)}$ and combine them to get $\dfrac{dM}{dt}$.

SOLUTION:

$$M = \sqrt{(H_1)^2 + (H_2)^2 + (H_3)^2} = \sqrt{(12\times 10^4)^2 + (-3\times 10^4)^2 + (4\times 10^4)^2} = 13\times 10^4$$

$$\frac{D(M)}{D(H_1,H_2,H_3)} = \left(\frac{\partial M}{\partial H_1}, \frac{\partial M}{\partial H_2}, \frac{\partial M}{\partial H_3}\right)$$

$$= \left(\frac{1}{2}\frac{2H_1}{\sqrt{(H_1)^2+(H_2)^2+(H_3)^2}}, \frac{1}{2}\frac{2H_2}{\sqrt{(H_1)^2+(H_2)^2+(H_3)^2}}, \frac{1}{2}\frac{2H_3}{\sqrt{(H_1)^2+(H_2)^2+(H_3)^2}}\right)$$

$$= \frac{1}{M}(H_1,H_2,H_3) = \frac{1}{13\times 10^4}(12\times 10^4, -3\times 10^4, 4\times 10^4) = \left(\frac{12}{13}, \frac{-3}{13}, \frac{4}{13}\right)$$

$$\frac{D(H_1,H_2,H_3)}{D(x,y,z)} = \begin{pmatrix} \frac{\partial H_1}{\partial x} & \frac{\partial H_1}{\partial y} & \frac{\partial H_1}{\partial z} \\ \frac{\partial H_2}{\partial x} & \frac{\partial H_2}{\partial y} & \frac{\partial H_2}{\partial z} \\ \frac{\partial H_3}{\partial x} & \frac{\partial H_3}{\partial y} & \frac{\partial H_3}{\partial z} \end{pmatrix} = \begin{pmatrix} 2 & -1 & 3 \\ 4 & 0 & -1 \\ -2 & 1 & 3 \end{pmatrix}$$

$$\frac{D(x,y,z)}{D(t)} = \begin{pmatrix} dx/dt \\ dy/dt \\ dz/dt \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$$

$$\frac{dM}{dt} = \frac{D(M)}{D(H_1,H_2,H_3)}\frac{D(H_1,H_2,H_3)}{D(x,y,z)}\frac{D(x,y,z)}{D(t)}$$

$$= \left(\frac{12}{13}, \frac{-3}{13}, \frac{4}{13}\right)\begin{pmatrix} 2 & -1 & 3 \\ 4 & 0 & -1 \\ -2 & 1 & 3 \end{pmatrix}\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} = \left(\frac{12}{13}, \frac{-3}{13}, \frac{4}{13}\right)\begin{pmatrix} 11 \\ 11 \\ -5 \end{pmatrix} = \frac{79}{13}$$

2. (20 points) Consider the vector space $(P_3)^2 = P_3 \times P_3$ of ordered pairs of polymonials of degree less than 3. For example, $(2+3x+4x^2, 5-4x-3x^2) \in (P_3)^2$. We take the "standard" basis to be:

$$\vec{e}_1 = (1,0), \quad \vec{e}_2 = (x,0), \quad \vec{e}_3 = (x^2,0), \quad \vec{e}_4 = (0,1), \quad \vec{e}_5 = (0,x), \quad \vec{e}_6 = (0,x^2)$$

and the alternate basis to be:

$$\vec{E}_1 = (1,0), \quad \vec{E}_2 = (1+x,0), \quad \vec{E}_3 = (1+x^2,0), \quad \vec{E}_4 = (0,1), \quad \vec{E}_5 = (0,1+x), \quad \vec{E}_6 = (0,1+x^2)$$

a. Find the change of basis matrices $\underset{E\leftarrow e}{C}$ and $\underset{e\leftarrow E}{C}$. Be sure to say which is which.

SOLUTION:
$$\begin{aligned}\vec{E}_1 &= \vec{e}_1\\ \vec{E}_2 &= \vec{e}_1+\vec{e}_2\\ \vec{E}_3 &= \vec{e}_1+\vec{e}_3\\ \vec{E}_4 &= \vec{e}_4\\ \vec{E}_5 &= \vec{e}_4+\vec{e}_5\\ \vec{E}_6 &= \vec{e}_4+\vec{e}_6\end{aligned} \qquad \underset{e\leftarrow E}{C} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 1 & 1\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\end{pmatrix}$$

Invert this matrix using row operations, or solve for the $\vec{e}$'s.

$$\begin{aligned}\vec{e}_1 &= \vec{E}_1\\ \vec{e}_2 &= \vec{E}_2-\vec{E}_1\\ \vec{e}_3 &= \vec{E}_3-\vec{E}_1\\ \vec{e}_4 &= \vec{E}_4\\ \vec{e}_5 &= \vec{E}_5-\vec{E}_4\\ \vec{e}_6 &= \vec{E}_6-\vec{E}_4\end{aligned} \qquad \underset{E\leftarrow e}{C} = \begin{pmatrix} 1 & -1 & -1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & -1 & -1\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\end{pmatrix}$$

b. Find the components $(\vec{v})_e$ of the vector $\vec{v} = (2+3x+4x^2, 5-4x-3x^2)$ relative to the e-basis.

SOLUTION: $\vec{v} = 2\vec{e}_1 + 3\vec{e}_2 + 4\vec{e}_3 + 5\vec{e}_4 - 4\vec{e}_5 - 3\vec{e}_6 \qquad (\vec{v})_e = (2,3,4,5,-4,-3)^T$

c. Use a change of basis matrix to find the components $(\vec{v})_E$ of the vector $\vec{v}$ relative to the E-basis.

SOLUTION: $$(\vec{v})_E = \begin{pmatrix} 1 & -1 & -1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & -1 & -1\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}2\\3\\4\\5\\-4\\-3\end{pmatrix} = \begin{pmatrix}-5\\3\\4\\12\\-4\\-3\end{pmatrix}$$

d. Check your answer to (c) by hooking the components $(\vec{v})_E$ onto the E-basis vectors and simplifying.

SOLUTION:

$$\begin{aligned}\vec{v} &= -5\vec{E}_1 + 3\vec{E}_2 + 4\vec{E}_3 + 12\vec{E}_4 - 4\vec{E}_5 - 3\vec{E}_6\\ &= -5(1,0) + 3(1+x,0) + 4(1+x^2,0) + 12(0,1) - 4(0,1+x) - 3(0,1+x^2)\\ &= (-5,0) + (3+3x,0) + (4+4x^2,0) + (0,12) + (0,-4-4x) + (0,-3-3x^2)\\ &= (2+3x+4x^2, 5-4x-3x^2)\end{aligned}$$

3. (20 points) Consider the subspace $V = \text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ of $\mathbb{R}^4$ with the inner product $\langle \vec{p}, \vec{q} \rangle = \vec{p}^{\mathsf{T}} G \vec{q}$ where $^{\mathsf{T}}$ means transpose and

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \vec{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad \text{and} \quad G = \begin{pmatrix} 2 & 2 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 3 \end{pmatrix}.$$

Find an orthonormal basis for V by applying the Gram-Schmidt procedure to the vectors $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$.

SOLUTION:

$$\vec{w}_1 = \vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \langle \vec{w}_1, \vec{w}_1 \rangle = \vec{w}_1^{\mathsf{T}} G \vec{w}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 2 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = 2$$

$$\vec{u}_1 = \frac{\vec{w}_1}{|\vec{w}_1|} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \Leftarrow$$

$$\langle \vec{v}_2, \vec{w}_1 \rangle = \vec{v}_2^{\mathsf{T}} G \vec{w}_1 = \begin{pmatrix} 1 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 2 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = 4$$

$$\vec{w}_2 = \vec{v}_2 - \frac{\langle \vec{v}_2, \vec{w}_1 \rangle}{\langle \vec{w}_1, \vec{w}_1 \rangle} \vec{w}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} - \frac{4}{2} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\langle \vec{w}_2, \vec{w}_2 \rangle = \vec{w}_2^{\mathsf{T}} G \vec{w}_2 = \begin{pmatrix} -1 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 2 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 3 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = 1$$

$$\vec{u}_2 = \frac{\vec{w}_2}{|\vec{w}_2|} = \frac{1}{\sqrt{1}} \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \Leftarrow$$

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$$\langle \vec{v}_3, \vec{w}_1 \rangle = \vec{v}_3^{\mathsf{T}} G \vec{w}_1 = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 2 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = 4$$

$$\langle \vec{v}_3, \vec{w}_2 \rangle = \vec{v}_3^{\mathsf{T}} G \vec{w}_2 = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 2 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 3 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = 1$$

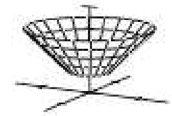
$$\vec{w}_3 = \vec{v}_3 - \frac{\langle \vec{v}_3, \vec{w}_1 \rangle}{\langle \vec{w}_1, \vec{w}_1 \rangle} \vec{w}_1 - \frac{\langle \vec{v}_3, \vec{w}_2 \rangle}{\langle \vec{w}_2, \vec{w}_2 \rangle} \vec{w}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \frac{4}{2} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} - \frac{1}{1} \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

$$\langle \vec{w}_3, \vec{w}_3 \rangle = \vec{w}_3^{\mathsf{T}} G \vec{w}_3 = \begin{pmatrix} 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 2 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 3 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} = 2$$

$$\vec{u}_3 = \frac{\vec{w}_3}{|\vec{w}_3|} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \quad \Leftarrow$$

4. (25 points) Verify Stokes' Theorem $\iint_C \vec{\nabla} \times \vec{F} \cdot d\vec{S} = \oint_{\partial C} \vec{F} \cdot d\vec{s}$

for the slice of the cone C given by $z = \sqrt{x^2 + y^2}$ for $1 \le z \le 3$.



oriented down and out, and the vector field $\vec{F} = (-yz, xz, z^2)$.

Note: The boundary of the cone has 2 pieces:

the top circle, $x^2 + y^2 = 9$, and the bottom circle, $x^2 + y^2 = 1$.

Be sure to check the orientations. Use the following steps:

a. The cone may be parametrized as $\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, r)$

Compute the surface integral by successively finding:

$$\vec{e}_r,\quad \vec{e}_\theta,\quad \vec{N},\quad \vec{\nabla} \times \vec{F},\quad \vec{\nabla} \times \vec{F}\Big|_{\vec{R}(r,\theta)},\quad \iint_C \vec{\nabla} \times \vec{F} \cdot d\vec{S}$$

$$\begin{array}{r|ccc|} & \hat{\imath} & \hat{\jmath} & \hat{k} \\ \vec{e}_r = & (\cos\theta & \sin\theta, & 1) \\ \vec{e}_\theta = & (-r\sin\theta, & r\cos\theta, & 0) \end{array} \qquad \begin{aligned} \vec{N} &= \vec{e}_r \times \vec{e}_\theta \\ &= \hat{\imath}(-r\cos\theta) - \hat{\jmath}(r\sin\theta) + \hat{k}(r\cos^2\theta + r\sin^2\theta) \\ &= (-r\cos\theta, -r\sin\theta, r) \end{aligned}$$

Reverse $\vec{N} = (r\cos\theta, r\sin\theta, -r)$ so it points down and out.

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -yz, & xz, & z^2 \end{vmatrix} = \hat{\imath}(0 - x) - \hat{\jmath}(0 - -y) + \hat{k}(z - -z) = (-x, -y, 2z)$$

$$\vec{\nabla} \times \vec{F}\Big|_{\vec{R}(r,\theta)} = (-r\cos\theta, -r\sin\theta, 2r)$$

$$\iint_C \vec{\nabla} \times \vec{F} \cdot d\vec{S} = \iint_C \vec{\nabla} \times \vec{F} \cdot \vec{N}\, dr\, d\theta = \int_0^{\pi} \int_1^3 (-r^2\cos^2\theta - r^2\sin^2\theta - 2r^2)\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_1^3 (-3r^2)\, dr\, d\theta = 2\pi\Big[-r^3\Big]_1^3 = -52\pi$$

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b. The top circle T may be parametrized as $\vec{r}(\theta) = (3\cos\theta, 3\sin\theta, 3)$.

Compute the line integral over the top circle by successively finding:

$$\vec{v}, \quad \vec{F}\Big|_{\vec{r}(\theta)}, \quad \oint_T \vec{F} \cdot d\vec{s}$$

$\vec{v} = (-3\sin\theta, 3\cos\theta, 0)$ Reverse $\vec{v} = (3\sin\theta, -3\cos\theta, 0)$

$\vec{F}\Big|_{\vec{r}(\theta)} = (-yz, xz, z^2) = (-9\sin\theta, 9\cos\theta, 9)$

$$\oint_T \vec{F} \cdot d\vec{s} = \int_0^{2\pi} \vec{F} \cdot \vec{v}\, d\theta = \int_0^{2\pi} (-27\sin^2\theta - 27\cos^2\theta)\, d\theta = -54\pi$$

c. Compute the line integral over the bottom circle by successively finding:

$$\vec{r}(\theta), \quad \vec{v}, \quad \vec{F}\Big|_{\vec{r}(\theta)}, \quad \oint_B \vec{F} \cdot d\vec{s}$$

$\vec{r}(\theta) = (\cos\theta, \sin\theta, 1)$ $\vec{v} = (-\sin\theta, \cos\theta, 0)$ Oriented correctly

$\vec{F}\Big|_{\vec{r}(\theta)} = (-yz, xz, z^2) = (-\sin\theta, \cos\theta, 1)$

$$\oint_B \vec{F} \cdot d\vec{s} = \int_0^{2\pi} \vec{F} \cdot \vec{v}\, d\theta = \int_0^{2\pi} (\sin^2\theta + \cos^2\theta)\, d\theta = 2\pi$$

d. Combine the results from (b) and (c) to get $\oint_{\partial C} \vec{F} \cdot d\vec{s}$.

$$\oint_{\partial C} \vec{F} \cdot d\vec{s} = \int_T \vec{F} \cdot d\vec{s} + \oint_B \vec{F} \cdot d\vec{s} = -54\pi + 2\pi = -52\pi$$

5. (15 points) Compute $\iint\limits_{H} \vec{F} \cdot d\vec{S}$ for the vector field $\vec{F} = (xy^2, yx^2, x^2 + y^2)$ over the hemisphere H given by $x^2 + y^2 + z^2 = 25$ with $z \ge 0$ oriented upward.

HINT: Use Gauss' Theorem to convert this surface integral into a volume integral over a solid hemisphere V and a surface integral over a disk D. Then add or subtract the answers to get the required integral. Be careful with the orientations.

SOLUTION: Gauss' Theorem says:

$$\iiint\limits_{V} \vec{\nabla} \cdot \vec{F}\, dV = \iint\limits_{\partial V} \vec{F} \cdot d\vec{S} = \iint\limits_{H\uparrow} \vec{F} \cdot d\vec{S} + \iint\limits_{D\downarrow} \vec{F} \cdot d\vec{S} = \iint\limits_{H\uparrow} \vec{F} \cdot d\vec{S} - \iint\limits_{D\uparrow} \vec{F} \cdot d\vec{S}$$

where H is oriented upward and D is initially oriented downward, but reversing its orientation, flips the sign. Then we solve for the desired integral:

$$\iint\limits_{H\uparrow} \vec{F} \cdot d\vec{S} = \iiint\limits_{V} \vec{\nabla} \cdot \vec{F}\, dV + \iint\limits_{D\uparrow} \vec{F} \cdot d\vec{S}$$

We compute the volume integral in spherical coordinates:

$$\vec{\nabla} \cdot \vec{F} = y^2 + x^2 + 0 = \rho^2 \sin^2\varphi \sin^2\theta + \rho^2 \sin^2\varphi \cos^2\theta = \rho^2 \sin^2\varphi \qquad dV = \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta$$

$$\iiint\limits_{V} \vec{\nabla} \cdot \vec{F}\, dV = \int_0^{2\pi} \int_0^{\pi/2} \int_0^{5} (\rho^2 \sin^2\varphi) \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta = 2\pi \left[\frac{\rho^5}{5} \right]_0^5 \int_0^{\pi/2} (1 - \cos^2\varphi) \sin\varphi\, d\varphi$$

$$= 2\pi 5^4 \int_0^{\pi/2} \sin\varphi - \cos^2\varphi \sin\varphi\, d\varphi = 1250\pi \left[-\cos\varphi + \frac{\cos^3\varphi}{3} \right]_0^{\pi/2} = 1250\pi \left[0 - \left(-1 + \frac{1}{3} \right) \right] = \frac{2500\pi}{3}$$

To compute the surface integral over the disk, we parametrize the disk as:
$\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, 0)$.

$$\begin{array}{r} \\ \vec{e}_r = \\ \vec{e}_\theta = \end{array} \left| \begin{array}{ccc} \hat{\imath} & \hat{\jmath} & \hat{k} \\ (\ \cos\theta & \sin\theta & 0) \\ (-r\sin\theta & r\cos\theta & 0) \end{array} \right| \qquad \vec{N} = (0, 0, r)$$

Oriented upward which is correct.

$$\vec{F}\Big|_{\vec{R}(r,\theta)} = (r^3 \cos\theta \sin^2\theta, r^3 \sin\theta \cos^2\theta, r^2) \qquad \vec{F} \cdot \vec{N} = r^3$$

$$\iint\limits_{D\uparrow} \vec{F} \cdot d\vec{S} = \int_0^{2\pi} \int_0^{5} r^3\, dr\, d\theta = 2\pi \left[\frac{r^4}{4} \right]_0^5 = \frac{625\pi}{2}$$

Combining the integrals, we have:

$$\iint\limits_{H\uparrow} \vec{F} \cdot d\vec{S} = \iiint\limits_{V} \vec{\nabla} \cdot \vec{F}\, dV + \iint\limits_{D\uparrow} \vec{F} \cdot d\vec{S} = \frac{2500\pi}{3} + \frac{625\pi}{2} = \frac{6875\pi}{6}$$

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SOLUTION 2: To compute the integral directly, we parametrize the hemisphere as

$$\vec{R}(\varphi,\theta) = (5\sin\varphi\cos\theta, 5\sin\varphi\sin\theta, 5\cos\varphi)$$

$$\begin{matrix} & \hat{\imath} & \hat{\jmath} & \hat{k} \\ \vec{e}_\varphi = & (5\cos\varphi\cos\theta & 5\cos\varphi\sin\theta & -5\sin\varphi) \\ \vec{e}_\theta = & (-5\sin\varphi\sin\theta & 5\sin\varphi\cos\theta & 0 \quad) \end{matrix}$$

$$\vec{N} = (5^2\sin^2\varphi\cos\theta, 5^2\sin^2\varphi\sin\theta, 5^2\sin\varphi\cos\varphi)$$

which is correctly oriented upward.

$$\vec{F}\Big|_{\vec{R}(\varphi,\theta)} = (xy^2, yx^2, x^2+y^2) = (5^3\sin^3\varphi\sin^2\theta\cos\theta, 5^3\sin^3\varphi\sin\theta\cos^2\theta, 5^2\sin^2\varphi)$$

$$\vec{F}\cdot\vec{N} = 5^5\sin^5\varphi\sin^2\theta\cos^2\theta + 5^5\sin^5\varphi\sin^2\theta\cos^2\theta + 5^4\sin^3\varphi\cos\varphi$$
$$= 2\cdot 5^5\sin^5\varphi\sin^2\theta\cos^2\theta + 5^4\sin^3\varphi\cos\varphi$$

$$\iint_{H\uparrow}\vec{F}\cdot d\vec{S} = \int_0^{2\pi}\int_0^{\pi/2}(2\cdot 5^5\sin^5\varphi\sin^2\theta\cos^2\theta + 5^4\sin^3\varphi\cos\varphi)\,d\varphi\,d\theta = \frac{6875\pi}{6}$$ after lots of work integrating.