Name________________________________

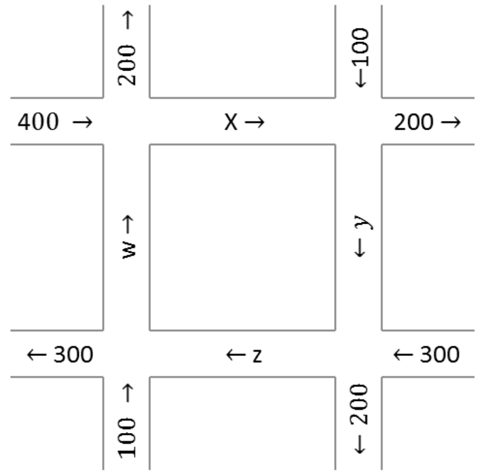
Math 311   Exam 1 Version A   Spring 2015

Section 502   Solutions   P. Yasskin

Points indicated. Show all work.

1	/30	5	/20
2	/10	6	/15
3	/10	7	/15
4	/10	Total	/110

1. (30 points) Solve the traffic flow system shown at the right. Find the smallest non-negative values of w, x, y, z. In your augmented matrix, keep the variables in the order w, x, y, z.



Solution: The equations are:

$$\begin{aligned} w+400 &= x+200 & \qquad w-x &= -200 \\ x+100 &= y+200 & \qquad x-y &= 100 \\ y+300 &= z+200 & \qquad y-z &= -100 \\ z+100 &= w+300 & \qquad -w+z &= 200 \end{aligned}$$

The augmented matrix and row operations are:

$$\left(\begin{array}{cccc|c} 1 & -1 & 0 & 0 & -200 \\ 0 & 1 & -1 & 0 & 100 \\ 0 & 0 & 1 & -1 & -100 \\ -1 & 0 & 0 & 1 & 200 \end{array}\right) \begin{array}{l} \\ \\ \\ R_4+R_1 \end{array} \Rightarrow \left(\begin{array}{cccc|c} 1 & -1 & 0 & 0 & -200 \\ 0 & 1 & -1 & 0 & 100 \\ 0 & 0 & 1 & -1 & -100 \\ 0 & -1 & 0 & 1 & 0 \end{array}\right) \begin{array}{l} R_1+R_2 \\ \\ \\ R_4+R_2 \end{array} \Rightarrow$$

$$\left(\begin{array}{cccc|c} 1 & 0 & -1 & 0 & -100 \\ 0 & 1 & -1 & 0 & 100 \\ 0 & 0 & 1 & -1 & -100 \\ 0 & 0 & -1 & 1 & 100 \end{array}\right) \begin{array}{l} R_1+R_3 \\ R_2+R_3 \\ \\ R_4+R_3 \end{array} \Rightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & -1 & -200 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & -100 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{array}{l} w = r-200 \\ x = r \\ y = r-100 \\ z = r \end{array}$$

For the smallest non-negative solution, we take $r = 200$:

$w = 0$ $x = 200$ $y = 100$ $z = 200$

2. (10 points) By definition, a matrix, A, is nilpotent with degree 2 if $A^2 = \mathbf{0}$.
Prove if A is nilpotent with degree 2, then $\mathbf{1} - A$ is non-singular and $(\mathbf{1} - A)^{-1} = \mathbf{1} + A$.

Solution: $(\mathbf{1} - A)(\mathbf{1} + A) = \mathbf{1} + A - A - A^2 = \mathbf{1} - A^2 = \mathbf{1}$ since $A^2 = \mathbf{0}$.
Thus $(\mathbf{1} - A)$ and $(\mathbf{1} + A)$ are inverses and $\mathbf{1} - A$ is invertible and non-singular.

3. (10 points) For an $n \times n$ matrix A, define its trace to be $tr(A) = \sum_{i=1}^{n} A_{ii}$ i.e. the sum of its diagonal entries. Prove, for $n \times n$ matrices A and B, $tr(AB) = tr(BA)$.

Solution: $tr(AB) = \sum_{i=1}^{n} (AB)_{ii} = \sum_{i=1}^{n}\sum_{k=1}^{n} A_{ik}B_{ki} = \sum_{i=1}^{n}\sum_{k=1}^{n} B_{ki}A_{ik} = \sum_{k=1}^{n}\sum_{i=1}^{n} B_{ki}A_{ik} = \sum_{k=1}^{n} (BA)_{kk} = tr(BA)$

4. (10 points) A matrix A satisfies $E_1E_2E_3A = B$ where

$$E_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 3 & * & * \\ 0 & -4 & 7 \\ 0 & 0 & -2 \end{pmatrix}$$

and the $*$'s represent unknown non-zero numbers. Find $\det A$.

Solution: $\det E_1 = -1 \quad \det E_2 = 3 \quad \det E_3 = 1 \quad \det B = (3)(-4)(-2) = 24$

$$\det A = \frac{\det B}{\det E_1 \det E_2 \det E_3} = \frac{24}{(-1)(3)(1)} = -8$$

5. (20 points) Sulfuric acid (H_2SO_4) combines with sodium hydroxide $(NaOH)$ to produce sodium sulfate (Na_2SO_4) and water (H_2O) according to the chemical equation:

$$a\,H_2SO_4 + b\,NaOH \rightarrow c\,Na_2SO_4 + d\,H_2O$$

To balance this chemical equation, you must solve the system

$$\begin{array}{lrl} H: & 2a+b & = 2d \\ S: & a & = c \\ O: & 4a+b & = 4c+d \\ Na: & b & = 2c \end{array}$$

a. Write out the augmented matrix for this system of 4 equations in 4 unknowns. DO NOT SOLVE.

Solution: $\left(\begin{array}{cccc|c} 2 & 1 & 0 & -2 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 4 & 1 & -4 & -1 & 0 \\ 0 & 1 & -2 & 0 & 0 \end{array}\right)$

b. Compute the determinant of the matrix of coefficients.

Solution: Use 2 row operations. Then expand on column 2:

$$\det A = \begin{vmatrix} 2 & 1 & 0 & -2 \\ 1 & 0 & -1 & 0 \\ 4 & 1 & -4 & -1 \\ 0 & 1 & -2 & 0 \end{vmatrix} \begin{array}{c} \\ \\ R_3 - R_1 \\ R_4 - R_1 \end{array} = \begin{vmatrix} 2 & 1 & 0 & -2 \\ 1 & 0 & -1 & 0 \\ 2 & 0 & -4 & 1 \\ -2 & 0 & -2 & 2 \end{vmatrix} = (-1)\begin{vmatrix} 1 & -1 & 0 \\ 2 & -4 & 1 \\ -2 & -2 & 2 \end{vmatrix}$$

Add column 1 to column 2. Then expand on row 1:

$$\det A = (-1)\begin{vmatrix} 1 & 0 & 0 \\ 2 & -2 & 1 \\ -2 & -4 & 2 \end{vmatrix} = (-1)(1)\begin{vmatrix} -2 & 1 \\ -4 & 2 \end{vmatrix} = (-1)(1)(-4+4) = 0$$

c. What property of the augmented matrix says there is at least one solution? (Circle one.)

- **i.** The fact that the determinant of the matrix of coefficients is zero.
- **ii.** The fact that the determinant of the matrix of coefficients is non-zero.
- **iii.** The fact that the matrix of coefficients is square (4×4).
- **iv.** The fact that the system is homogeneous (the right hand sides are all zero). CORRECT

d. What additional property says there are infinitely many solutions? (Circle one.)

- **i.** The fact that the determinant of the matrix of coefficients is zero. CORRECT
- **ii.** The fact that the determinant of the matrix of coefficients is non-zero.
- **iii.** The fact that the matrix of coefficients is square (4×4).
- **iv.** The fact that the system is homogeneous (the right hand sides are all zero).

6. $\left(15 \text{ points}\right)$ Let $A = \begin{pmatrix} 1 & 2 & a \\ 4 & 3 & b \\ c & d & 0 \end{pmatrix}$. Given that $\det(A) = 5$, determine each of the following:

$$\begin{vmatrix} 1 & 2 & a \\ 4+c & 3+d & b \\ c & d & 0 \end{vmatrix} = 5 \qquad \begin{vmatrix} 1 & 4 & a \\ 4 & 6 & b \\ c & 2d & 0 \end{vmatrix} = 10 \qquad \begin{vmatrix} 4 & 3 & b \\ 1 & 2 & a \\ c & d & 0 \end{vmatrix} = -5$$

$\det(2A) = 40$ $\qquad \det(A^{-1}) = \frac{1}{5}$

7. $\left(15 \text{ points}\right)$ For each of the following sets with operations, determine whether it forms a vector space. If it does, just say "Yes". If it does not, say "No" and give an axiom or other property it violates and show why.

a. The set of all power series centered at 3, $S = \left\{ a = \sum_{n=0}^{\infty} a_n (x-3)^n \right\}$ with

$a \oplus b = \sum_{n=0}^{\infty} (a_n + b_n)(x-3)^n$ and $\alpha \odot a = \sum_{n=0}^{\infty} \alpha\, a_n (x-3)^n$

Solution: Yes, it is a vector space.

b. $F_{\text{even}}[-1,1] = \{f : [-1,1] \to \mathbb{R} \mid f(-x) = f(x)\}$ with
$(f \oplus g)(x) = f(-x) + g(-x)$ and $(\alpha \odot f)(x) = \alpha f(-x)$

Solution: Yes, it is a vector space since
$(f \oplus g)(x) = f(-x) + g(-x) = f(x) + g(x)$ and $(\alpha \odot f)(x) = \alpha f(-x) = \alpha f(x)$

c. $F_{\text{odd}}[-1,1] = \{f : [-1,1] \to \mathbb{R} \mid f(-x) = -f(x)\}$ with
$(f \oplus g)(x) = f(-x) + g(-x)$ and $(\alpha \odot f)(x) = \alpha f(-x)$

Solution: No, it violates A_8 since $(1 \odot f)(x) = f(-x) = -f(x) \neq f(x)$