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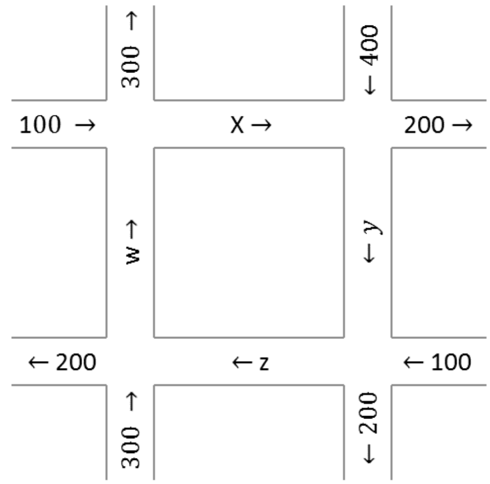
Math 311 Exam 1 Version B Spring 2015
Section 502 Solutions P. Yasskin

Points indicated. Show all work.

1	/30	5	/20
2	/10	6	/15
3	/10	7	/15
4	/10	Total	/110

1. (30 points) Solve the traffic flow system shown at the right. Find the smallest non-negative values of w, x, y, z. In your augmented matrix, keep the variables in the order w, x, y, z.



Solution: The equations are:

$w + 100 = x + 300 \qquad w - x = 200$

$x + 400 = y + 200 \qquad x - y = -200$

$y + 100 = z + 200 \qquad y - z = 100$

$z + 300 = w + 200 \qquad -w + z = -100$

The augmented matrix and row operations are:

$$\left(\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 200 \\ 0 & 1 & -1 & 0 & -200 \\ 0 & 0 & 1 & -1 & 100 \\ -1 & 0 & 0 & 1 & -100 \end{array}\right) \begin{array}{l} \\ \\ \\ R_4 + R_1 \end{array} \Rightarrow \left(\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 200 \\ 0 & 1 & -1 & 0 & -200 \\ 0 & 0 & 1 & -1 & 100 \\ 0 & -1 & 0 & 1 & 100 \end{array}\right) \begin{array}{l} R_1 + R_2 \\ \\ \\ R_4 + R_2 \end{array} \Rightarrow$$

$$\left(\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & -200 \\ 0 & 0 & 1 & -1 & 100 \\ 0 & 0 & -1 & 1 & -100 \end{array}\right) \begin{array}{l} R_1 + R_3 \\ R_2 + R_3 \\ \\ R_4 + R_3 \end{array} \Rightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 100 \\ 0 & 1 & 0 & -1 & -100 \\ 0 & 0 & 1 & -1 & 100 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{array}{l} w = r + 100 \\ x = r - 100 \\ y = r + 100 \\ z = r \end{array}$$

For the smallest non-negative solution, we take $r = 100$:

$w = 200 \qquad x = 0 \qquad y = 200 \qquad z = 100$

2. $\left(10\text{ points}\right)$ By definition, a matrix, A, is idempotent if $A^2 = A$.
Prove if A is idempotent, then $\mathbf{1} + A$ is non-singular and $(\mathbf{1} + A)^{-1} = \mathbf{1} - \frac{1}{2}A$.

Solution: $(\mathbf{1} + A)\left(\mathbf{1} - \frac{1}{2}A\right) = \mathbf{1} - \frac{1}{2}A + A - \frac{1}{2}A^2 = \mathbf{1} - \frac{1}{2}A + A - \frac{1}{2}A = \mathbf{1}$ since $A^2 = A$.
Thus $(\mathbf{1} + A)$ and $\left(\mathbf{1} - \frac{1}{2}A\right)$ are inverses and $\mathbf{1} + A$ is invertible and non-singular.

3. $\left(10\text{ points}\right)$ If A is a 50×60 matrix while B and C are 60×80 matrices, prove $A(B + C) = AB + AC$.

HINT: Prove equality of the ij-component of each side.

Solution: $[A(B+C)]_{ij} = \sum_{k=1}^{60} A_{ik}(B+C)_{kj} = \sum_{k=1}^{60} A_{ik}(B_{kj} + C_{kj}) = \sum_{k=1}^{60} A_{ik}B_{kj} + \sum_{k=1}^{60} A_{ik}C_{kj}$

$= [AB]_{ij} + [AC]_{ij} = [AB + AC]_{ij}$

These are equal. So $A(B + C) = AB + AC$

4. $\left(10\text{ points}\right)$ A matrix A satisfies $E_1E_2E_3A = B$ where

$$E_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \quad E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} -5 & 3 & * \\ 0 & 2 & * \\ 0 & 0 & -2 \end{pmatrix}$$

and the $*$'s represent unknown non-zero numbers. Find $\det A$.

Solution: $\det E_1 = -1 \qquad \det E_2 = 2 \qquad \det E_3 = 1 \qquad \det B = (-5)(2)(-2) = 20$

$\det A = \dfrac{\det B}{\det E_1 \det E_2 \det E_3} = \dfrac{20}{(-1)(2)(1)} = -10$

5. (20 points) Phosphoric acid (H_3PO_4) combines with sodium hydroxide ($NaOH$) to produce trisodium phosphate (Na_3PO_4) and water (H_2O) according to the chemical equation:

$$a\,H_3PO_4 + b\,NaOH \rightarrow c\,Na_3PO_4 + d\,H_2O$$

To balance this chemical equation, you must solve the system

$$\begin{aligned} H:&\quad 3a+b &= 2d \\ P:&\quad a &= c \\ O:&\quad 4a+b &= 4c+d \\ Na:&\quad b &= 3c \end{aligned}$$

a. Write out the augmented matrix for this system of 4 equations in 4 unknowns. DO NOT SOLVE.

Solution: $\left(\begin{array}{cccc|c} 3 & 1 & 0 & -2 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 4 & 1 & -4 & -1 & 0 \\ 0 & 1 & -3 & 0 & 0 \end{array}\right)$

b. Compute the determinant of the matrix of coefficients.

Solution: Use 2 row operations. Then expand on column 2:

$$\det A = \begin{vmatrix} 3 & 1 & 0 & -2 \\ 1 & 0 & -1 & 0 \\ 4 & 1 & -4 & -1 \\ 0 & 1 & -3 & 0 \end{vmatrix} \begin{matrix} \\ \\ R_3 - R_1 \\ R_4 - R_1 \end{matrix} = \begin{vmatrix} 3 & 1 & 0 & -2 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & -4 & 1 \\ -3 & 0 & -3 & 2 \end{vmatrix} = (-1)\begin{vmatrix} 1 & -1 & 0 \\ 1 & -4 & 1 \\ -3 & -3 & 2 \end{vmatrix}$$

Add column 1 to column 2. Then expand on row 1:

$$\det A = (-1)\begin{vmatrix} 1 & 0 & 0 \\ 1 & -3 & 1 \\ -3 & -6 & 2 \end{vmatrix} = (-1)(1)\begin{vmatrix} -3 & 1 \\ -6 & 2 \end{vmatrix} = (-1)(1)(-6+6) = 0$$

c. What property of the augmented matrix says there is at least one solution? (Circle one.)

- **i.** The fact that the determinant of the matrix of coefficients is non-zero.
- **ii.** The fact that the determinant of the matrix of coefficients is zero.
- **iii.** The fact that the system is homogeneous (the right hand sides are all zero). CORRECT
- **iv.** The fact that the matrix of coefficients is square (4×4).

d. What additional property says there are infinitely many solutions? (Circle one.)

- **i.** The fact that the determinant of the matrix of coefficients is non-zero.
- **ii.** The fact that the determinant of the matrix of coefficients is zero. CORRECT
- **iii.** The fact that the system is homogeneous (the right hand sides are all zero).
- **iv.** The fact that the matrix of coefficients is square (4×4).

6. (15 points) Let $A = \begin{pmatrix} 1 & 2 & a \\ 4 & 3 & b \\ c & d & 0 \end{pmatrix}$. Given that $\det(A) = 4$, determine each of the following:

$\begin{vmatrix} 1+c & 2+d & a \\ 4 & 3 & b \\ c & d & 0 \end{vmatrix} = 4$ $\quad$ $\begin{vmatrix} 1 & 6 & a \\ 4 & 9 & b \\ c & 3d & 0 \end{vmatrix} = 12$ $\quad$ $\begin{vmatrix} 2 & 1 & a \\ 3 & 4 & b \\ d & c & 0 \end{vmatrix} = -4$

$\det(3A) = 108$ $\quad$ $\det(A^{-1}) = \dfrac{1}{4}$

7. (15 points) For each of the following sets with operations, determine whether it forms a vector space. If it does, just say "Yes". If it does not, say "No" and give an axiom or other property it violates and show why.

a. The set of infinite sequences, $S = \{a = [a_1, a_2, \cdots, a_n, \cdots]\}$, with $a \oplus b = [a_1 + b_1, a_2 + b_2, \cdots, a_n + b_n, \cdots]$ and $\alpha \odot a = [\alpha a_1, \alpha a_2, \cdots, \alpha a_n, \cdots]$

Solution: Yes, it is a vector space.

b. The set of traceless 2×2 matrices $M_0(2,2) = \left\{ A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \;\middle|\; A_{11} + A_{22} = 0 \right\}$ with

$A \oplus B = \begin{pmatrix} A_{11} + B_{11} & A_{12} + B_{12} \\ A_{21} + B_{21} & A_{22} + B_{22} \end{pmatrix}$ and $\alpha \odot A = \begin{pmatrix} \alpha A_{22} & \alpha A_{12} \\ \alpha A_{21} & \alpha A_{11} \end{pmatrix}$

Solution: No, it violates A_8 since $1 \odot A = \begin{pmatrix} A_{22} & A_{12} \\ A_{21} & A_{11} \end{pmatrix} \neq A$

c. The set of traceless 2×2 matrices $M_0(2,2) = \left\{ A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \;\middle|\; A_{11} + A_{22} = 0 \right\}$

$A \oplus B = \begin{pmatrix} A_{11} + B_{22} & A_{12} + B_{12} \\ A_{21} + B_{21} & A_{22} + B_{11} \end{pmatrix}$ and $\alpha \odot A = \begin{pmatrix} \alpha A_{11} & \alpha A_{12} \\ \alpha A_{21} & \alpha A_{22} \end{pmatrix}$

Solution: No, it violates A_1 since $B \oplus A = \begin{pmatrix} B_{11} + A_{22} & B_{12} + A_{12} \\ B_{21} + A_{21} & B_{22} + A_{11} \end{pmatrix} \neq A \oplus B$