Name________________________________

Math 311 Exam 2 Version A Spring 2015

Section 503 P. Yasskin

Points indicated. Show all work.

1	/15	4	/27
2	/38	5 E.C.	/10
3	/25	Total	/115

1. (15 points) Let P_5 be the vector space of polynomials of degree less than 5.
Consider the subspace $V = Span(v_1, v_2, v_3, v_4)$ where
$v_1 = 2 + 3x^2$, $v_2 = x - 3x^3$, $v_3 = 2 - x + 3x^4$, $v_4 = x^2 + x^3 - x^4$
Find a basis for V What is $\dim V$?

2. $\left(38 \text{ points}\right)$ Consider the vector space $V = Span\left(\sin^2(x), \cos^2(x), \sin(x)\cos(x)\right)$ with the usual addition and scalar multiplication of functions. Two bases are:

$e_1 = \sin^2(x) \quad e_2 = \cos^2(x) \quad e_3 = \sin(x)\cos(x)$ and $E_1 = 1 \quad E_2 = \sin(2x) \quad E_3 = \cos(2x)$

Note: You do NOT need to prove they are bases.

Hints: Here are some useful identities:

$$\sin(2x) = 2\sin(x)\cos(x), \quad \cos(2x) = \cos^2(x) - \sin^2(x), \quad \sin^2(x) = \frac{1-\cos(2x)}{2}, \quad \cos^2(x) = \frac{1+\cos(2x)}{2}$$

a. (5) Find the change of basis matrix $\underset{E \leftarrow e}{C}$ from the e basis to the E basis by using the identities.

b. (5) Find the change of basis matrix $\underset{e \leftarrow E}{C}$ from the E basis to the e basis by using the identities.

c. (2) Verify $\underset{E \leftarrow e}{C}$ and $\underset{e \leftarrow E}{C}$ are inverses.

d. (4) For the function $f = \cos(2x) + 4\cos^2(x)$, what are its components $(f)_e$ and $(f)_E$?

e. (5) Find the matrix $\underset{e \leftarrow e}{A}$ of the derivative operator $D = \frac{d}{dx}$ relative to the e basis.

f. (5) Find the matrix $\underset{E \leftarrow E}{B}$ of the derivative operator $D = \frac{d}{dx}$ relative to the E basis.
Do NOT use the change of basis matrices.

g. (2) A and B are related by a similarity transformation: $B = SAS^{-1}$. What is S?

h. (3) What is $\mathrm{Im}(D)$? Give a basis. What is $\dim(\mathrm{Im}(D))$?
HINT: Let $f = a \cdot 1 + b \cdot \sin(2x) + c \cdot \cos(2x)$.

i. (3) What is $Ker(D)$? Give a basis. What is $\dim(Ker(D))$?

j. (2) Is D onto? Why or why not?

k. (2) Is D 1-1? Why or why not?

3. (25 points) Consider the vector space S of symmetric 2×2 matrices. The following are symmetric, biinear forms on S. Which one(s) are inner products? Why or why not? You do not need to check they are symmetic or bilinear, just that they are positive definite.

HINTS: Let $A = \begin{pmatrix} a & b \\ b & d \end{pmatrix}$. Compute $\langle A, A \rangle$. Look for perfect squares or complete the squares.

a. (9) $\langle A, B \rangle = tr(AGB^{\mathsf{T}})$ where $G = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

b. (8) $\langle A, B \rangle = tr(AGB^{\mathsf{T}})$ where $G = \begin{pmatrix} 4 & 1 \\ 1 & 1 \end{pmatrix}$

c. (8) $\langle A, B \rangle = tr(AGB^{\mathsf{T}})$ where $G = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$

4. (27 points) Consider the vector space S of symmetric 2×2 matrices with the inner product

$$\langle A, B \rangle = tr(AGB^{\mathsf{T}}) \qquad \text{where } G = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}.$$

a. (8) Find the angle between the matrices $A = \begin{pmatrix} 4 & 3 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 4 \\ 4 & 3 \end{pmatrix}$.

b. (19) A basis for S is $V_1 = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$ $V_2 = \begin{pmatrix} 3 & 2 \\ 2 & 2 \end{pmatrix}$ $V_3 = \begin{pmatrix} 5 & 0 \\ 0 & -5 \end{pmatrix}$.

Apply the Gram-Schmidt Procedure to the (V_1, V_2, V_3) basis to produce an orthogonal basis (W_1, W_2, W_3) and an orthonormal basis (U_1, U_2, U_3).

5. $\left(10 \text{ points EC}\right)$ Consider the vector space $V = (\mathbb{R}^+)^2 = \left\{(x_1, x_2) \mid x_1 > 0 \text{ and } x_2 > 0\right\}$ consisting of ordered pairs of positive numbers with addition and multiplication defined by

$$(x_1, x_2) \oplus (y_1, y_2) = (x_1 y_1, x_2 y_2) \text{ and } a \odot (x_1, x_2) = \left(x_1^a, x_2^a\right)$$

So vector addition is real number multiplication of corresponding components and scalar multiplication is real number exponentiation of each component. Note the zero vector is $\vec{0} = (1, 1)$.

a. (5) Is $u_1 = (1, 3)$ and $u_2 = (2, 1)$ a basis? Why or why not?

b. (5) Is $v_1 = (1, 1)$ and $v_2 = (2, 1)$ a basis? Why or why not?