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Math 311 Exam 2 Version A Spring 2015
Section 503 Solutions P. Yasskin

Points indicated. Show all work.

1	/15	4	/27
2	/38	5 E.C.	/10
3	/25	Total	/115

1. (15 points) Let P_5 be the vector space of polynomials of degree less than 5.
Consider the subspace $V = Span(v_1, v_2, v_3, v_4)$ where

$$v_1 = 2 + 3x^2, \quad v_2 = x - 3x^3, \quad v_3 = 2 - x + 3x^4, \quad v_4 = x^2 + x^3 - x^4$$

Find a basis for V What is $\dim V$?

Solution: To see if v_1, v_2, v_3, v_4 are linearly independent, we write:

$av_1 + bv_2 + cv_3 + dv_4 = 0$ and solve for a, b, c, d.

$$a(2+3x^2) + b(x - 3x^3) + c(2 - x + 3x^4) + d(x^2 + x^3 - x^4) = 0$$

$$(2a + 2c) + (b - c)x + (3a + d)x^2 + (-3b + d)x^3 + (3c - d)x^4 = 0$$

$$2a + 2c = 0 \qquad b - c = 0 \qquad 3a + d = 0 \qquad -3b + d = 0 \qquad 3c - d = 0$$

$$\left(\begin{array}{cccc|c} 2 & 0 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 3 & 0 & 0 & 1 & 0 \\ 0 & -3 & 0 & 1 & 0 \\ 0 & 0 & 3 & -1 & 0 \end{array}\right) \begin{array}{l} \frac{1}{2}R_1 \\ \\ \\ \\ \\ \end{array} \Rightarrow \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 3 & 0 & 0 & 1 & 0 \\ 0 & -3 & 0 & 1 & 0 \\ 0 & 0 & 3 & -1 & 0 \end{array}\right) \begin{array}{l} \\ \\ R_3 - 3R_1 \\ \\ \\ \end{array}$$

$$\Rightarrow \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & -3 & 1 & 0 \\ 0 & -3 & 0 & 1 & 0 \\ 0 & 0 & 3 & -1 & 0 \end{array}\right) \begin{array}{l} \\ \\ \\ R_4 + 3R_2 \\ \\ \end{array} \Rightarrow \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & -3 & 1 & 0 \\ 0 & 0 & -3 & 1 & 0 \\ 0 & 0 & 3 & -1 & 0 \end{array}\right) \begin{array}{l} \\ \\ -\frac{1}{3}R_3 \\ \\ \\ \end{array}$$

$$\Rightarrow \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -\frac{1}{3} & 0 \\ 0 & 0 & -3 & 1 & 0 \\ 0 & 0 & 3 & -1 & 0 \end{array}\right) \begin{array}{l} R_1 - R_3 \\ R_2 + R_3 \\ \\ R_4 + 3R_3 \\ R_5 - 3R_3 \end{array} \Rightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & \frac{1}{3} & 0 \\ 0 & 1 & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 1 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$\Rightarrow a = -d/3 \qquad b = d/3 \qquad c = d/3$ Pick $d = 1$. So $a = -1/3 \qquad b = 1/3 \qquad c = 1/3$

So they are not linearly independent. Further,

$-\frac{1}{3}v_1 + \frac{1}{3}v_2 + \frac{1}{3}v_3 + v_4 = 0$ or $v_4 = \frac{1}{3}v_1 - \frac{1}{3}v_2 - \frac{1}{3}v_3$

So v_1, v_2, v_3 also span and the above computation without column 4 shows they are linearly independent. So v_1, v_2, v_3 is a basis and $\dim V = 3$.

2. (38 points) Consider the vector space $V = Span\left(\sin^2(x), \cos^2(x), \sin(x)\cos(x)\right)$ with the usual addition and scalar multiplication of functions. Two bases are:

$e_1 = \sin^2(x) \quad e_2 = \cos^2(x) \quad e_3 = \sin(x)\cos(x)$ and $E_1 = 1 \quad E_2 = \sin(2x) \quad E_3 = \cos(2x)$

Note: You do NOT need to prove they are bases.

Hints: Here are some useful identities:

$$\sin(2x) = 2\sin(x)\cos(x), \quad \cos(2x) = \cos^2(x) - \sin^2(x), \quad \sin^2(x) = \frac{1-\cos(2x)}{2}, \quad \cos^2(x) = \frac{1+\cos(2x)}{2}$$

a. (5) Find the change of basis matrix $\underset{E\leftarrow e}{C}$ from the e basis to the E basis by using the identities.

$$\begin{aligned} e_1 &= \sin^2(x) &&= \frac{1-\cos(2x)}{2} &&= \frac{1}{2}E_1 - \frac{1}{2}E_3 \\ e_2 &= \cos^2(x) &&= \frac{1+\cos(2x)}{2} &&= \frac{1}{2}E_1 + \frac{1}{2}E_3 \\ e_3 &= \sin(x)\cos(x) &&= \frac{\sin(2x)}{2} &&= \frac{1}{2}E_2 \end{aligned} \quad \Rightarrow \underset{E\leftarrow e}{C} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$$

b. (5) Find the change of basis matrix $\underset{e\leftarrow E}{C}$ from the E basis to the e basis by using the identities.

$$\begin{aligned} E_1 &= 1 &&= \sin^2(x) + \cos^2(x) &&= e_1 + e_2 \\ E_2 &= \sin(2x) &&= 2\sin(x)\cos(x) &&= 2e_3 \\ E_3 &= \cos(2x) &&= \cos^2(x) - \sin^2(x) &&= -e_1 + e_2 \end{aligned} \quad \Rightarrow \underset{e\leftarrow E}{C} = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 0 & 1 \\ 0 & 2 & 0 \end{pmatrix}$$

c. (2) Verify $\underset{E\leftarrow e}{C}$ and $\underset{e\leftarrow E}{C}$ are inverses.

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & -1 \\ 1 & 0 & 1 \\ 0 & 2 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}+\frac{1}{2} & 0 & -\frac{1}{2}+\frac{1}{2} \\ 0 & \frac{1}{2}\cdot 2 & 0 \\ -\frac{1}{2}+\frac{1}{2} & 0 & \frac{1}{2}+\frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

d. (4) For the function $f = \cos(2x) + 4\cos^2(x)$, what are its components $(f)_e$ and $(f)_E$?

$$f = \cos(2x) + 4\cos^2(x) = \cos^2(x) - \sin^2(x) + 4\cos^2(x) = -1e_1 + 5e_2 \qquad (f)_e = \begin{pmatrix} -1 \\ 5 \\ 0 \end{pmatrix}$$

$$f = \cos(2x) + 4\cos^2(x) = \cos(2x) + 2(1+\cos(2x)) = 2E_1 + 3E_3 \qquad (f)_E = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}$$

e. (5) Find the matrix $\underset{e \leftarrow e}{A}$ of the derivative operator $D = \frac{d}{dx}$ relative to the e basis.

$$\begin{aligned} D(e_1) &= D(\sin^2(x)) &&= 2\sin(x)\cos(x) &&= 2e_3 \\ D(e_2) &= D(\cos^2(x)) &&= -2\sin(x)\cos(x) &&= -2e_3 \\ D(e_3) &= D(\sin(x)\cos(x)) &&= \cos^2(x) - \sin^2(x) &&= -e_1 + e_2 \end{aligned} \qquad \underset{e \leftarrow e}{A} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \\ 2 & -2 & 0 \end{pmatrix}$$

f. (5) Find the matrix $\underset{E \leftarrow E}{B}$ of the derivative operator $D = \frac{d}{dx}$ relative to the E basis.
Do NOT use the change of basis matrices.

$$\begin{aligned} D(E_1) &= D(1) &&= 0 &&= 0 \\ D(E_2) &= D(\sin(2x)) &&= 2\cos(2x) &&= 2E_3 \\ D(E_3) &= D(\cos(2x)) &&= -2\sin(2x) &&= -2E_2 \end{aligned} \qquad \underset{E \leftarrow E}{B} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 2 & 0 \end{pmatrix}$$

g. (2) A and B are related by a similarity transformation: $B = SAS^{-1}$. What is S?

$\underset{E \leftarrow E}{B} = \underset{E \leftarrow e}{C}\ \underset{e \leftarrow e}{A}\ \underset{e \leftarrow E}{C}$ So $S = \underset{E \leftarrow e}{C}$

h. (3) What is $\text{Im}(D)$? Give a basis. What is $\dim(\text{Im}(D))$?
HINT: Let $f = a \cdot 1 + b \cdot \sin(2x) + c \cdot \cos(2x)$.

$D(f) = 2b \cdot \cos(2x) - 2c \cdot \sin(2x)$
$\text{Im}(D) = \{D(f)\} = \{2b \cdot \cos(2x) - 2c \cdot \sin(2x)\} = Span(\cos(2x), \sin(2x)) = Span(E_2, E_3)$
Basis: E_2, E_3 $\quad \dim(\text{Im}(D)) = 2$

i. (3) What is $Ker(D)$? Give a basis. What is $\dim(Ker(D))$?

$Ker(D) = \{f \mid D(f) = 0\} = \{f = C \text{ where } C \text{ is a constant}\} = Span(1) = Span(E_1)$
Basis: E_1 $\quad \dim(Ker(D)) = 1$

j. (2) Is D onto? Why or why not?

D is not onto because $\text{Im}(D) = Span(\cos(2x), \sin(2x)) \neq V = Span(1, \sin(2x), \cos(2x))$

k. (2) Is D 1-1? Why or why not?

D is not 1-1 because $Ker(D) = Span(E_1) \neq \{0\}$

3. (25 points) Consider the vector space S of symmetric 2×2 matrices. The following are symmetric, biinear forms on S. Which one(s) are inner products? Why or why not? You do not need to check they are symmetic or bilinear, just that they are positive definite.

HINTS: Let $A = \begin{pmatrix} a & b \\ b & d \end{pmatrix}$. Compute $\langle A, A \rangle$. Look for perfect squares or complete the squares.

a. (9) $\langle A, B \rangle = tr(AGB^{\mathsf{T}})$ where $G = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

$$\langle A, A \rangle = tr\left(\begin{pmatrix} a & b \\ b & d \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ b & d \end{pmatrix}\right) = tr\left(\begin{pmatrix} a+b & a+b \\ b+d & b+d \end{pmatrix}\begin{pmatrix} a & b \\ b & d \end{pmatrix}\right)$$

$$= tr\left(\begin{pmatrix} (a+b)^2 & (a+b)(b+d) \\ (b+d)(a+b) & (b+d)^2 \end{pmatrix}\right) = (a+b)^2 + (b+d)^2 \geq 0$$

If $\langle A, A \rangle = 0$ then $(a+b)^2 + (b+d)^2 = 0$. So $a = -b$ and $d = -b$, but nothing says $b = 0$.

So if $A = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$ then $\langle A, A \rangle = 0$ but $A \neq \mathbf{0}$. So this is not an inner product.

b. (8) $\langle A, B \rangle = tr(AGB^{\mathsf{T}})$ where $G = \begin{pmatrix} 4 & 1 \\ 1 & 1 \end{pmatrix}$

$$\langle A, A \rangle = tr\left(\begin{pmatrix} a & b \\ b & d \end{pmatrix}\begin{pmatrix} 4 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ b & d \end{pmatrix}\right) = tr\left(\begin{pmatrix} 4a+b & a+b \\ 4b+d & b+d \end{pmatrix}\begin{pmatrix} a & b \\ b & d \end{pmatrix}\right)$$

$$= tr\left(\begin{pmatrix} 4a^2 + 2ab + b^2 & (4a+b)b + (a+b)d \\ (4b+d)a + (b+d)b & 4b^2 + 2bd + d^2 \end{pmatrix}\right) = 3a^2 + (a+b)^2 + 3b^2 + (b+d)^2 \geq 0$$

If $\langle A, A \rangle = 0$ then $3a^2 + (a+b)^2 + 3b^2 + (b+d)^2 = 0$. So $a = 0$, $b = 0$, $d = -b = 0$.
So this is an inner product.

c. (8) $\langle A, B \rangle = tr(AGB^{\mathsf{T}})$ where $G = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$

$$\langle A, A \rangle = tr\left(\begin{pmatrix} a & b \\ b & d \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ b & d \end{pmatrix}\right) = tr\left(\begin{pmatrix} a+3b & 3a+b \\ b+3d & 3b+d \end{pmatrix}\begin{pmatrix} a & b \\ b & d \end{pmatrix}\right)$$

$$= tr\left(\begin{pmatrix} a^2 + 6ab + b^2 & (a+3b)b + (3a+b)d \\ (b+3d)a + (3b+d)b & b^2 + 6bd + d^2 \end{pmatrix}\right) = (a+3b)^2 - 8b^2 + (b+3d)^2 - 8d^2$$

This may not be positive, for example if $a = -3b$ and $b = -3d$ but $b \neq 0$ then $\langle A, A \rangle = -8b^2 - 8d^2 < 0$.
So this is not an inner product.

4. (27 points) Consider the vector space S of symmetric 2×2 matrices with the inner product

$$\langle A, B \rangle = tr(AGB^{\mathsf{T}}) \qquad \text{where } G = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}.$$

a. (8) Find the angle between the matrices $A = \begin{pmatrix} 4 & 3 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 4 \\ 4 & 3 \end{pmatrix}$.

$$\langle A, B \rangle = tr\left(\begin{pmatrix} 4 & 3 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} 3 & 4 \\ 4 & 3 \end{pmatrix}\right) = tr\left(\begin{pmatrix} 4 & 12 \\ 3 & 16 \end{pmatrix}\begin{pmatrix} 3 & 4 \\ 4 & 3 \end{pmatrix}\right)$$

$$= tr\begin{pmatrix} 60 & 52 \\ 73 & 60 \end{pmatrix} = 120$$

$$\langle A, A \rangle = tr\left(\begin{pmatrix} 4 & 3 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} 4 & 3 \\ 3 & 4 \end{pmatrix}\right) = tr\left(\begin{pmatrix} 4 & 12 \\ 3 & 16 \end{pmatrix}\begin{pmatrix} 4 & 3 \\ 3 & 4 \end{pmatrix}\right)$$

$$= tr\begin{pmatrix} 52 & 60 \\ 60 & 73 \end{pmatrix} = 125$$

$$\langle B, B \rangle = tr\left(\begin{pmatrix} 3 & 4 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} 3 & 4 \\ 4 & 3 \end{pmatrix}\right) = tr\left(\begin{pmatrix} 3 & 16 \\ 4 & 12 \end{pmatrix}\begin{pmatrix} 3 & 4 \\ 4 & 3 \end{pmatrix}\right)$$

$$= tr\begin{pmatrix} 73 & 60 \\ 60 & 52 \end{pmatrix} = 125$$

$$\cos(\theta) = \frac{\langle A, B \rangle}{|A||B|} = \frac{120}{\sqrt{125}^{\,2}} = \frac{120}{125} = \frac{24}{25} \qquad \theta = \arccos\left(\frac{24}{25}\right)$$

b. (19) A basis for S is $V_1 = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$ $V_2 = \begin{pmatrix} 3 & 2 \\ 2 & 2 \end{pmatrix}$ $V_3 = \begin{pmatrix} 5 & 0 \\ 0 & -5 \end{pmatrix}$.

Apply the Gram-Schmidt Procedure to the (V_1, V_2, V_3) basis to produce an orthogonal basis (W_1, W_2, W_3) and an orthonormal basis (U_1, U_2, U_3).

$$W_1 = V_1 = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$$

$$\langle W_1, W_1 \rangle = tr\left(\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}\right) = tr\left(\begin{pmatrix} 3 & 0 \\ 0 & 8 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}\right)$$

$$= tr\begin{pmatrix} 9 & 0 \\ 0 & 16 \end{pmatrix} = 25 \qquad |W_1| = 5$$

$$\boxed{U_1 = \frac{1}{5}W_1 = \frac{1}{5}\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 3/5 & 0 \\ 0 & 2/5 \end{pmatrix}}$$

$$\langle V_2, W_1\rangle = tr\left(\begin{pmatrix} 3 & 2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}\right) = tr\left(\begin{pmatrix} 3 & 8 \\ 2 & 8 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}\right)$$

$$= tr\begin{pmatrix} 9 & 16 \\ 6 & 16 \end{pmatrix} = 25$$

$$W_2 = V_2 - \frac{\langle V_2, W_1\rangle}{\langle W_1, W_1\rangle} W_1 = \begin{pmatrix} 3 & 2 \\ 2 & 2 \end{pmatrix} - \frac{25}{25}\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$$

$$\langle W_2, W_2\rangle = tr\left(\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}\right) = tr\left(\begin{pmatrix} 0 & 8 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}\right)$$

$$= tr\left(\begin{pmatrix} 16 & 0 \\ 0 & 4 \end{pmatrix}\right) = 20 \qquad |W_2| = 2\sqrt{5}$$

$$\boxed{U_2 = \frac{1}{2\sqrt{5}} W_2 = \frac{1}{2\sqrt{5}}\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1/\sqrt{5} \\ 1/\sqrt{5} & 0 \end{pmatrix}}$$

$$\langle V_3, W_1\rangle = tr\left(\begin{pmatrix} 5 & 0 \\ 0 & -5 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}\right) = tr\left(\begin{pmatrix} 5 & 0 \\ 0 & -20 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}\right)$$

$$= tr\begin{pmatrix} 15 & 0 \\ 0 & -40 \end{pmatrix} = -25$$

$$\langle V_3, W_2\rangle = tr\left(\begin{pmatrix} 5 & 0 \\ 0 & -5 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}\right) = tr\left(\begin{pmatrix} 5 & 0 \\ 0 & -20 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}\right)$$

$$= tr\begin{pmatrix} 0 & 10 \\ -40 & 0 \end{pmatrix} = 0$$

$$W_3 = V_3 - \frac{\langle V_3, W_1\rangle}{\langle W_1, W_1\rangle} W_1 - \frac{\langle V_3, W_2\rangle}{\langle W_2, W_2\rangle} W_2 = \begin{pmatrix} 5 & 0 \\ 0 & -5 \end{pmatrix} - \frac{-25}{25}\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 8 & 0 \\ 0 & -3 \end{pmatrix}$$

$$\langle W_3, W_3\rangle = tr\left(\begin{pmatrix} 8 & 0 \\ 0 & -3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} 8 & 0 \\ 0 & -3 \end{pmatrix}\right) = tr\left(\begin{pmatrix} 8 & 0 \\ 0 & -12 \end{pmatrix}\begin{pmatrix} 8 & 0 \\ 0 & -3 \end{pmatrix}\right)$$

$$= tr\begin{pmatrix} 64 & 0 \\ 0 & 36 \end{pmatrix} = 100 \qquad |W_3| = 10$$

$$\boxed{U_3 = \frac{1}{10} W_3 = \frac{1}{10}\begin{pmatrix} 8 & 0 \\ 0 & -3 \end{pmatrix} = \begin{pmatrix} 4/5 & 0 \\ 0 & -3/10 \end{pmatrix}}$$

5. $\left(10 \text{ points EC}\right)$ Consider the vector space $V = (\mathbb{R}^+)^2 = \left\{(x_1, x_2) \mid x_1 > 0 \text{ and } x_2 > 0\right\}$ consisting of ordered pairs of positive numbers with addition and multiplication defined by

$$(x_1, x_2) \oplus (y_1, y_2) = (x_1 y_1, x_2 y_2) \text{ and } a \odot (x_1, x_2) = \left(x_1^{\,a}, x_2^{\,a}\right)$$

So vector addition is real number multiplication of corresponding components and scalar multiplication is real number exponentiation of each component. Note the zero vector is $\vec{0} = (1,1)$.

a. (5) Is $u_1 = (1,3)$ and $u_2 = (2,1)$ a basis? Why or why not?

Solution: To see if they span, we need to find a and b so that

$(x_1, x_2) = a \odot u_1 \oplus b \odot u_2$

But then

$(x_1, x_2) = a \odot (1,3) \oplus b \odot (2,1) = (1^a, 3^a) \oplus (2^b, 1^b) = (12^b, 3^a 1^b) = (2^b, 3^a)$

So $b = \log_2 x_1$ and $a = \log_3 x_2$.

To see if they are linearly independent, we write

$a \odot u_1 \oplus b \odot u_2 = \vec{0}$ and must see if $a = b = 0$ is the only solution. But this says

$(2^b, 3^a) = (1,1)$ So $b = \log_2 1 = 0$ and $a = \log_3 1 = 0$

So they span and are linearly independent and so are a basis.

b. (5) Is $v_1 = (1,1)$ and $v_2 = (2,1)$ a basis? Why or why not?

Solution: To see if they span, we need to find a and b so that

$(x_1, x_2) = a \odot v_1 \oplus b \odot v_2 = a \odot (1,1) \oplus b \odot (2,1) = (1^a, 1^a) \oplus (2^b, 1^b) = (1^a 2^b, 1^a 1^b) = (2^b, 1)$

So $b = \log_2 x_1$ and $x_2 = 1$. So there is no way to produce a vector with $x_2 \neq 1$.

They don't span!

To see if they are linearly independent, we write

$a \odot v_1 \oplus b \odot v_2 = \vec{0}$ and must see if $a = b = 0$ is the only solution. But this says

$(2^b, 1) = (1,1)$ So $b = \log_2 1 = 0$ but a can be arbitrary.

They are not linearly independent!

Either one says they are not a basis.