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Math 311 Exam 2 Version B Spring 2015
 Section 503 Solutions P. Yasskin

1	/15	4	/27
2	/38	5 E.C.	/10
3	/25	Total	/115

Points indicated. Show all work.

1. (15 points) Let P_5 be the vector space of polynomials of degree less than 5.

Consider the subspace $V = \text{Span}(v_1, v_2, v_3, v_4)$ where

$$v_1 = 2 + 3x^2, \quad v_2 = x - 3x^3, \quad v_3 = x^2 + x^3 - x^4, \quad v_4 = 2 - x + 3x^4$$

Find a basis for V . What is $\dim V$?

Solution: To see if v_1, v_2, v_3, v_4 are linearly independent, we write:

$$av_1 + bv_2 + cv_3 + dv_4 = 0 \text{ and solve for } a, b, c, d.$$

$$a(2 + 3x^2) + b(x - 3x^3) + c(x^2 + x^3 - x^4) + d(2 - x + 3x^4) = 0$$

$$(2a + 2d) + (b - d)x + (3a + c)x^2 + (-3b + c)x^3 + (-c + 3d)x^4 = 0$$

$$2a + 2d = 0 \quad b - d = 0 \quad 3a + c = 0 \quad -3b + d = 0 \quad -c + 3d = 0$$

$$\Rightarrow \left(\begin{array}{cccc|c} 2 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 3 & 0 & 1 & 0 & 0 \\ 0 & -3 & 1 & 0 & 0 \\ 0 & 0 & -1 & 3 & 0 \end{array} \right) \xrightarrow{\frac{1}{2}R_1} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 3 & 0 & 1 & 0 & 0 \\ 0 & -3 & 1 & 0 & 0 \\ 0 & 0 & -1 & 3 & 0 \end{array} \right) \xrightarrow{R_3 - 3R_1}$$

$$\Rightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & -3 & 1 & 0 & 0 \\ 0 & 0 & -1 & 3 & 0 \end{array} \right) \xrightarrow{R_4 + 3R_2} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & -1 & 3 & 0 \end{array} \right) \xrightarrow{\begin{array}{l} -\frac{1}{3}R_3 \\ R_4 - R_3 \\ R_5 + R_3 \end{array}}$$

$$\Rightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

$$\Rightarrow a = -d \quad b = d \quad c = 3d \quad \text{Pick } d = 1. \text{ So } a = -1 \quad b = 1 \quad c = 3$$

So they are not linearly independent. Further,

$$-v_1 + v_2 + 3v_3 + v_4 = 0 \quad \text{or} \quad v_4 = v_1 - v_2 - 3v_3$$

So v_1, v_2, v_3 also span and the above computation without column 4 shows they are linearly independent. So v_1, v_2, v_3 is a basis and $\dim V = 3$.

2. (38 points) Consider the vector space $V = \text{Span}(1, \sin(2x), \cos(2x))$ with the usual addition and scalar multiplication of functions. Two bases are:

$$e_1 = 1 \quad e_2 = \sin(2x) \quad e_3 = \cos(2x) \quad \text{and} \quad E_1 = \sin^2(x) \quad E_2 = \cos^2(x) \quad E_3 = \sin(x)\cos(x)$$

Note: You do NOT need to prove they are bases.

Hints: Here are some useful identities:

$$\sin(2x) = 2\sin(x)\cos(x), \quad \cos(2x) = \cos^2(x) - \sin^2(x), \quad \sin^2(x) = \frac{1 - \cos(2x)}{2}, \quad \cos^2(x) = \frac{1 + \cos(2x)}{2}$$

- a. (5) Find the change of basis matrix $C_{E \leftarrow e}$ from the e basis to the E basis by using the identities.

$$\begin{aligned} e_1 = 1 &= \sin^2(x) + \cos^2(x) = E_1 + E_2 \\ e_2 = \sin(2x) &= 2\sin(x)\cos(x) = 2E_3 \\ e_3 = \cos(2x) &= \cos^2(x) - \sin^2(x) = -E_1 + E_2 \end{aligned} \Rightarrow C_{E \leftarrow e} = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 0 & 1 \\ 0 & 2 & 0 \end{pmatrix}$$

- b. (5) Find the change of basis matrix $C_{e \leftarrow E}$ from the E basis to the e basis by using the identities.

$$\begin{aligned} E_1 = \sin^2(x) &= \frac{1 - \cos(2x)}{2} = \frac{1}{2}e_1 - \frac{1}{2}e_3 \\ E_2 = \cos^2(x) &= \frac{1 + \cos(2x)}{2} = \frac{1}{2}e_1 + \frac{1}{2}e_3 \\ E_3 = \sin(x)\cos(x) &= \frac{\sin(2x)}{2} = \frac{1}{2}e_2 \end{aligned} \Rightarrow C_{e \leftarrow E} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$$

- c. (2) Verify $C_{E \leftarrow e}$ and $C_{e \leftarrow E}$ are inverses.

$$\begin{pmatrix} 1 & 0 & -1 \\ 1 & 0 & 1 \\ 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} + \frac{1}{2} & \frac{1}{2} - \frac{1}{2} & 0 \\ \frac{1}{2} - \frac{1}{2} & \frac{1}{2} + \frac{1}{2} & 0 \\ 0 & 0 & 2 \cdot \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- d. (4) For the function $f = \sin(2x) + 4\sin^2(x)$, what are its components $(f)_e$ and $(f)_E$?

$$f = \sin(2x) + 4\sin^2(x) = \sin(2x) + 2(1 - \cos(2x)) = 2e_1 + e_2 - 2e_3 \quad (f)_e = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$$

$$f = \sin(2x) + 4\sin^2(x) = 2\sin(x)\cos(x) + 4\sin^2(x) = 4E_1 + 2E_3 \quad (f)_E = \begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix}$$

- e. (5) Find the matrix A of the derivative operator $D = \frac{d}{dx}$ relative to the e basis.

$$\begin{aligned} D(e_1) &= D(1) = 0 = 0 \\ D(e_2) &= D(\sin(2x)) = 2\cos(2x) = 2e_3 \\ D(e_3) &= D(\cos(2x)) = -2\sin(2x) = -2e_2 \end{aligned} \quad A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 2 & 0 \end{pmatrix}$$

- f. (5) Find the matrix B of the derivative operator $D = \frac{d}{dx}$ relative to the E basis.
Do NOT use the change of basis matrices.

$$\begin{aligned} D(E_1) &= D(\sin^2(x)) = 2\sin(x)\cos(x) = 2E_3 \\ D(E_2) &= D(\cos^2(x)) = -2\sin(x)\cos(x) = -2E_3 \\ D(E_3) &= D(\sin(x)\cos(x)) = \cos^2(x) - \sin^2(x) = -E_1 + E_2 \end{aligned} \quad B = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \\ 2 & -2 & 0 \end{pmatrix}$$

- g. (2) A and B are related by a similarity transformation: $B = SAS^{-1}$. What is S ?

$$B = C A C^{-1} \quad \text{So } S = C$$

$E \leftarrow E$ $E \leftarrow e$ $e \leftarrow e$ $e \leftarrow E$ $E \leftarrow e$

- h. (3) What is $\text{Im}(D)$? Give a basis. What is $\dim(\text{Im}(D))$?

HINT: Let $f = a \cdot 1 + b \cdot \sin(2x) + c \cdot \cos(2x)$.

$$D(f) = 2b \cdot \cos(2x) - 2c \cdot \sin(2x)$$

$$\text{Im}(D) = \{D(f)\} = \{2b \cdot \cos(2x) - 2c \cdot \sin(2x)\} = \text{Span}(\cos(2x), \sin(2x)) = \text{Span}(e_2, e_3)$$

$$\text{Basis: } e_2, e_3 \quad \dim(\text{Im}(D)) = 2$$

- i. (3) What is $\text{Ker}(D)$? Give a basis. What is $\dim(\text{Ker}(D))$?

$$\text{Ker}(D) = \{f \mid D(f) = 0\} = \{f = C \text{ where } C \text{ is a constant}\} = \text{Span}(1) = \text{Span}(e_1)$$

$$\text{Basis: } e_1 \quad \dim(\text{Ker}(D)) = 1$$

- j. (2) Is D onto? Why or why not?

$$D \text{ is not onto because } \text{Im}(D) = \text{Span}(\cos(2x), \sin(2x)) \neq V = \text{Span}(1, \sin(2x), \cos(2x))$$

- k. (2) Is D 1-1? Why or why not?

$$D \text{ is not 1-1 because } \text{Ker}(D) = \text{Span}(e_1) \neq \{0\}$$

3. (25 points) The following are symmetric, bilinear forms on $M(2,2)$, the vector space of 2×2 matrices. Which one(s) are inner products? Why or why not? You do not need to check they are symmetric or bilinear, just that they are positive definite.

HINTS: Let $A = \begin{pmatrix} a & b \\ b & d \end{pmatrix}$. Compute $\langle A, A \rangle$. Look for perfect squares or complete the squares.

a. (9) $\langle A, B \rangle = \text{tr}(AGB^T)$ where $G = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

$$\begin{aligned} \langle A, A \rangle &= \text{tr} \left(\begin{pmatrix} a & b \\ b & d \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ b & d \end{pmatrix} \right) = \text{tr} \left(\begin{pmatrix} a+b & a+b \\ b+d & b+d \end{pmatrix} \begin{pmatrix} a & b \\ b & d \end{pmatrix} \right) \\ &= \text{tr} \left(\begin{pmatrix} (a+b)^2 & (a+b)(b+d) \\ (b+d)(a+b) & (b+d)^2 \end{pmatrix} \right) = (a+b)^2 + (b+d)^2 \geq 0 \end{aligned}$$

If $\langle A, A \rangle = 0$ then $(a+b)^2 + (b+d)^2 = 0$. So $a = -b$ and $d = -b$, but nothing says $b = 0$.

So if $A = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$ then $\langle A, A \rangle = 0$ but $A \neq \mathbf{0}$. So this is not an inner product.

b. (8) $\langle A, B \rangle = \text{tr}(AGB^T)$ where $G = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}$

$$\begin{aligned} \langle A, A \rangle &= \text{tr} \left(\begin{pmatrix} a & b \\ b & d \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ b & d \end{pmatrix} \right) = \text{tr} \left(\begin{pmatrix} 3a+b & a+b \\ 3b+d & b+d \end{pmatrix} \begin{pmatrix} a & b \\ b & d \end{pmatrix} \right) \\ &= \text{tr} \left(\begin{pmatrix} 3a^2 + 2ab + b^2 & (3a+b)b + (a+b)d \\ (3b+d)a + (b+d)b & 3b^2 + 2bd + d^2 \end{pmatrix} \right) = 2a^2 + (a+b)^2 + 2b^2 + (b+d)^2 \geq 0 \end{aligned}$$

If $\langle A, A \rangle = 0$ then $2a^2 + (a+b)^2 + 2b^2 + (b+d)^2 = 0$. So $a = 0$, $b = 0$, $d = -b = 0$.

So this is an inner product.

c. (8) $\langle A, B \rangle = \text{tr}(AGB^T)$ where $G = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$

$$\begin{aligned} \langle A, A \rangle &= \text{tr} \left(\begin{pmatrix} a & b \\ b & d \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ b & d \end{pmatrix} \right) = \text{tr} \left(\begin{pmatrix} a+2b & 2a+b \\ b+2d & 2b+d \end{pmatrix} \begin{pmatrix} a & b \\ b & d \end{pmatrix} \right) \\ &= \text{tr} \left(\begin{pmatrix} a^2 + 4ab + b^2 & (a+2b)b + (2a+b)d \\ (b+2d)a + (2b+d)b & b^2 + 4bd + d^2 \end{pmatrix} \right) = (a+2b)^2 - 3b^2 + (b+2d)^2 - 3d^2 \end{aligned}$$

This may not be positive, for example if $a = -2b$ and $b = -2d$ but $b \neq 0$ then $\langle A, A \rangle = -3b^2 - 3d^2 < 0$.

So this is not an inner product.

4. (27 points) Consider the vector space S of symmetric 2×2 matrices with the inner product

$$\langle A, B \rangle = \text{tr}(AGB^T) \quad \text{where } G = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}.$$

a. (8) Find the angle between the matrices $A = \begin{pmatrix} 3 & 4 \\ 4 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & 3 \\ 3 & 4 \end{pmatrix}$.

$$\begin{aligned} \langle A, B \rangle &= \text{tr} \left(\begin{pmatrix} 3 & 4 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 4 & 3 \\ 3 & 4 \end{pmatrix} \right) = \text{tr} \left(\begin{pmatrix} 12 & 4 \\ 16 & 3 \end{pmatrix} \begin{pmatrix} 4 & 3 \\ 3 & 4 \end{pmatrix} \right) \\ &= \text{tr} \begin{pmatrix} 60 & 52 \\ 73 & 60 \end{pmatrix} = 120 \end{aligned}$$

$$\begin{aligned} \langle A, A \rangle &= \text{tr} \left(\begin{pmatrix} 3 & 4 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 4 \\ 4 & 3 \end{pmatrix} \right) = \text{tr} \left(\begin{pmatrix} 12 & 4 \\ 16 & 3 \end{pmatrix} \begin{pmatrix} 3 & 4 \\ 4 & 3 \end{pmatrix} \right) \\ &= \text{tr} \begin{pmatrix} 52 & 60 \\ 60 & 73 \end{pmatrix} = 125 \end{aligned}$$

$$\begin{aligned} \langle B, B \rangle &= \text{tr} \left(\begin{pmatrix} 4 & 3 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 4 & 3 \\ 3 & 4 \end{pmatrix} \right) = \text{tr} \left(\begin{pmatrix} 16 & 3 \\ 12 & 4 \end{pmatrix} \begin{pmatrix} 4 & 3 \\ 3 & 4 \end{pmatrix} \right) \\ &= \text{tr} \begin{pmatrix} 73 & 60 \\ 60 & 52 \end{pmatrix} = 125 \end{aligned}$$

$$\cos(\theta) = \frac{\langle A, B \rangle}{|A||B|} = \frac{120}{\sqrt{125}^2} = \frac{120}{125} = \frac{24}{25} \quad \theta = \arccos\left(\frac{24}{25}\right)$$

b. (19) A basis for S is $V_1 = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ $V_2 = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$ $V_3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

Apply the Gram-Schmidt Procedure to the (V_1, V_2, V_3) basis to produce an orthogonal basis (W_1, W_2, W_3) and an orthonormal basis (U_1, U_2, U_3) .

$$W_1 = V_1 = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

$$\begin{aligned} \langle W_1, W_1 \rangle &= \text{tr} \left(\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \right) = \text{tr} \left(\begin{pmatrix} 8 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \right) \\ &= \text{tr} \begin{pmatrix} 16 & 0 \\ 0 & 9 \end{pmatrix} = 25 \quad |W_1| = 5 \end{aligned}$$

$$U_1 = \frac{1}{5} W_1 = \frac{1}{5} \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 2/5 & 0 \\ 0 & 3/5 \end{pmatrix}$$

$$\begin{aligned}\langle V_2, W_1 \rangle &= \text{tr} \left(\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \right) = \text{tr} \left(\begin{pmatrix} 8 & 1 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \right) \\ &= \text{tr} \begin{pmatrix} 16 & 3 \\ 8 & 9 \end{pmatrix} = 25\end{aligned}$$

$$W_2 = V_2 - \frac{\langle V_2, W_1 \rangle}{\langle W_1, W_1 \rangle} W_1 = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} - \frac{25}{25} \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{aligned}\langle W_2, W_2 \rangle &= \text{tr} \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) = \text{tr} \left(\begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) \\ &= \text{tr} \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} = 5 \quad |W_2| = \sqrt{5}\end{aligned}$$

$$U_2 = \frac{1}{\sqrt{5}} W_2 = \frac{1}{\sqrt{5}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1/\sqrt{5} \\ 1/\sqrt{5} & 0 \end{pmatrix}$$

$$\begin{aligned}\langle V_3, W_1 \rangle &= \text{tr} \left(\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \right) = \text{tr} \left(\begin{pmatrix} -4 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \right) \\ &= \text{tr} \begin{pmatrix} -8 & 0 \\ 0 & 3 \end{pmatrix} = -5\end{aligned}$$

$$\begin{aligned}\langle V_3, W_2 \rangle &= \text{tr} \left(\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) = \text{tr} \left(\begin{pmatrix} -4 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) \\ &= \text{tr} \begin{pmatrix} 0 & -4 \\ 1 & 0 \end{pmatrix} = 0\end{aligned}$$

$$W_3 = V_3 - \frac{\langle V_3, W_1 \rangle}{\langle W_1, W_1 \rangle} W_1 - \frac{\langle V_3, W_2 \rangle}{\langle W_2, W_2 \rangle} W_2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{-5}{25} \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} -3/5 & 0 \\ 0 & 8/5 \end{pmatrix}$$

$$\begin{aligned}\langle W_3, W_3 \rangle &= \text{tr} \left(\frac{1}{5} \begin{pmatrix} -3 & 0 \\ 0 & 8 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \frac{1}{5} \begin{pmatrix} -3 & 0 \\ 0 & 8 \end{pmatrix} \right) = \frac{1}{25} \text{tr} \left(\begin{pmatrix} -12 & 0 \\ 0 & 8 \end{pmatrix} \begin{pmatrix} -3 & 0 \\ 0 & 8 \end{pmatrix} \right) \\ &= \frac{1}{25} \text{tr} \begin{pmatrix} 36 & 0 \\ 0 & 64 \end{pmatrix} = \frac{100}{25} = 4 \quad |W_3| = 2\end{aligned}$$

$$U_3 = \frac{1}{2} W_3 = \frac{1}{2} \begin{pmatrix} -3/5 & 0 \\ 0 & 8/5 \end{pmatrix} = \begin{pmatrix} -3/10 & 0 \\ 0 & 4/5 \end{pmatrix}$$

5. (10 points EC) Consider the vector space $V = (\mathbb{R}^+)^2 = \{(x_1, x_2) \mid x_1 > 0 \text{ and } x_2 > 0\}$ consisting of ordered pairs of positive numbers with addition and multiplication defined by

$$(x_1, x_2) \oplus (y_1, y_2) = (x_1 y_1, x_2 y_2) \text{ and } a \odot (x_1, x_2) = (x_1^a, x_2^a)$$

So vector addition is real number multiplication of corresponding components and scalar multiplication is real number exponentiation of each component. Note the zero vector is $\vec{0} = (1, 1)$.

- a. (5) Is $u_1 = (1, 2)$ and $u_2 = (3, 1)$ a basis? Why or why not?

Solution: To see if they span, we need to find a and b so that

$$(x_1, x_2) = a \odot u_1 \oplus b \odot u_2$$

But then

$$(x_1, x_2) = a \odot (1, 2) \oplus b \odot (3, 1) = (1^a, 2^a) \oplus (3^b, 1^b) = (1^a 3^b, 2^a 1^b) = (3^b, 2^a)$$

$$\text{So } b = \log_3 x_1 \text{ and } a = \log_2 x_2.$$

To see if they are linearly independent, we write

$$a \odot u_1 \oplus b \odot u_2 = \vec{0} \text{ and must see if } a = b = 0 \text{ is the only solution. But this says } (3^b, 2^a) = (1, 1) \text{ So } b = \log_3 1 = 0 \text{ and } a = \log_2 1 = 0$$

So they span and are linearly independent and so are a basis.

- b. (5) Is $v_1 = (1, 1)$ and $v_2 = (3, 1)$ a basis? Why or why not?

Solution: To see if they span, we need to find a and b so that

$$(x_1, x_2) = a \odot v_1 \oplus b \odot v_2 = a \odot (1, 1) \oplus b \odot (3, 1) = (1^a, 1^a) \oplus (3^b, 1^b) = (1^a 3^b, 1^a 1^b) = (3^b, 1)$$

$$\text{So } b = \log_3 x_1 \text{ and } x_2 = 1. \text{ So there is no way to produce a vector with } x_2 \neq 1.$$

They don't span!

To see if they are linearly independent, we write

$$a \odot v_1 \oplus b \odot v_2 = \vec{0} \text{ and must see if } a = b = 0 \text{ is the only solution. But this says } (3^b, 1) = (1, 1) \text{ So } b = \log_3 1 = 0 \text{ but } a \text{ can be arbitrary.}$$

They are not linearly independent!

Either one says they are not a basis.