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Math 311 Exam 3 Version A Spring 2015
Section 502 Solutions P. Yasskin

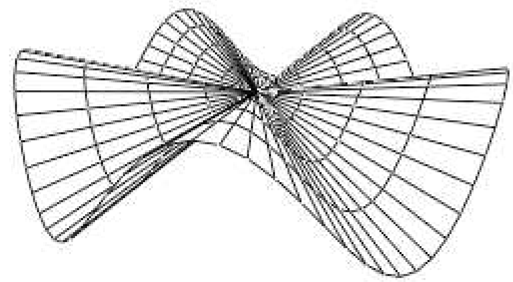
Points indicated. Show all work.

1	/20	3	/30
2	/36	4	/26
		Total	/112

1. (20 points) Compute $\iint\limits_S \vec\nabla \times \vec F \cdot d\vec S$ for $\vec F = (-y, x, z)$ over the "clam shell" surface, S, parametrized by

$$\vec R(r,\theta) = (r\cos\theta, r\sin\theta, r\sin(5\theta))$$

for $r \le 3$ oriented upward.



HINTS: Use Stokes Theorem.
What is the value of r on the boundary?

Stokes Theorem says $\iint\limits_S \vec\nabla \times \vec F \cdot d\vec S = \int\limits_{\partial S} \vec F \cdot d\vec s$ where ∂S is the boundary curve.

Since $r = 3$ on the boundary, the boundary curve is $\vec r(\theta) = \vec R(3,\theta) = (3\cos\theta, 3\sin\theta, 3\sin(5\theta))$

The velocity is $\vec v = (-3\sin\theta, 3\cos\theta, 15\cos(5\theta))$

The vector field on the boundary is $\vec F(\vec r(\theta)) = (-y, x, z) = (-3\sin\theta, 3\cos\theta, 3\sin(5\theta))$

$\vec F \cdot \vec v = 9\sin^2\theta + 9\cos^2\theta + 45\sin(5\theta)\cos(5\theta) = 9 + 45\sin(5\theta)\cos(5\theta)$

$$\int\int \vec\nabla \times \vec F \cdot d\vec S = \int \vec F \cdot d\vec s = \int_0^{2\pi} \vec F \cdot \vec v\, d\theta = \int_0^{2\pi} 9 + 45\sin(5\theta)\cos(5\theta)\, d\theta = \left[9\theta + \frac{9}{2}\sin^2(5\theta)\right]_0^{2\pi} = 18\pi$$

This can also be done directly without using Stokes Theorem.

2. (36 points) Let $V = Span(e^{2x}+e^{-2x}, e^{2x}-e^{-2x})$ be the vector space of functions spanned by the basis

$$e_1 = e^{2x}+e^{-2x}, \qquad e_2 = e^{2x}-e^{-2x}$$

Consider the linear operator $L : V \to V$ given by $L(f) = 3\frac{df}{dx}$. Our goals are to compute the eigenvalues and eigenfunctions of the linear operator L, to find the similarity transformation which diagonalizes the matrix of L and use this similarity transformation to compute a matrix power.

a. (5 pts) Find the matrix of L relative to the (e_1, e_2) basis. Call it $\underset{e\leftarrow e}{A}$.

$$L(e_1) = L(e^{2x}+e^{-2x}) = 3\frac{d(e^{2x}+e^{-2x})}{dx} = 6e^{2x}-6e^{-2x} = 6e_2$$

$$L(e_2) = L(e^{2x}-e^{-2x}) = 3\frac{d(e^{2x}-e^{-2x})}{dx} = 6e^{2x}+6e^{-2x} = 6e_1$$

$$\underset{e\leftarrow e}{A} = \begin{pmatrix} 0 & 6 \\ 6 & 0 \end{pmatrix}$$

b. (3 pts) Find the characteristic polynomial for $\underset{e\leftarrow e}{A}$.
Factor it and identify the eigenvalues of $\underset{e\leftarrow e}{A}$. These are also the eigenvalues of L.

$$\det(A-\lambda\mathbf{1}) = \begin{vmatrix} -\lambda & 6 \\ 6 & -\lambda \end{vmatrix} = \lambda^2 - 36 = (\lambda+6)(\lambda-6) \qquad \lambda = -6, 6$$

c. (8 pts) Find the eigenvector(s) of $\underset{e\leftarrow e}{A}$ for each eigenvalue, as vectors in $\mathbb{R}^2$.
Name them $\vec{v}_1$ and $\vec{v}_2$.

$$\lambda = -6: \quad \left(\begin{array}{cc|c} 6 & 6 & 0 \\ 6 & 6 & 0 \end{array}\right) \Rightarrow \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \qquad \vec{v}_1 = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -r \\ r \end{pmatrix} \qquad \vec{v}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\lambda = 6: \quad \left(\begin{array}{cc|c} -6 & 6 & 0 \\ 6 & -6 & 0 \end{array}\right) \Rightarrow \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right) \qquad \vec{v}_2 = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} r \\ r \end{pmatrix} \qquad \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

d. (6 pts) Convert the eigenvectors of $\underset{e\leftarrow e}{A}$ into eigenfunctions of L as functions in V.
Name them f_1 and f_2 and simplify them.
Then compute $L(f_1)$ and $L(f_2)$ to verify f_1 and f_2 are eigenfunctions.

Hint: Remember that the components of $\vec{v}_1$ and $\vec{v}_2$ are components of f_1 and f_2 relative to the (e_1, e_2) basis.

$$f_1 = -1e_1 + 1e_2 = -(e^{2x}+e^{-2x}) + (e^{2x}-e^{-2x}) = -2e^{-2x}$$

$$f_2 = 1e_1 + 1e_2 = (e^{2x}+e^{-2x}) + (e^{2x}-e^{-2x}) = 2e^{2x}$$

$$L(f_1) = 3\frac{d(-2e^{-2x})}{dx} = 12e^{-2x} = -6(-2e^{-2x}) = -6f_1$$

$$L(f_{21}) = 3\frac{d(2e^{2x})}{dx} = 12e^{2x} = 6(2e^{2x}) = 6f_2$$

e. (3 pts) Using the eigenfunctions as a new (f_1, f_2) basis for V, find the matrix of L relative to the (f_1, f_2) basis. Call it $\underset{f\leftarrow f}{D}$.

Since (f_1, f_2) is a basis of eigenvectors, the matrix of L relative to the (f_1, f_2) basis will be diagonal and the diagonal entries will be the eigenvalues.

$$\underset{f\leftarrow f}{D} = \begin{pmatrix} -6 & 0 \\ 0 & 6 \end{pmatrix}$$

f. (5 pts) Find the change of basis matrices $\underset{e\leftarrow f}{C}$ and $\underset{f\leftarrow e}{C}$ between the (e_1, e_2) basis to the (f_1, f_2) bases. Be sure to identify which is which.

$$f_1 = -1e_1 + 1e_2 \qquad f_2 = 1e_1 + 1e_2$$

$$\underset{e\leftarrow f}{C} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \qquad \underset{f\leftarrow e}{C} = \left(\underset{e\leftarrow f}{C}\right)^{-1} = \frac{1}{-2}\begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

g. (2 pts) A and D are related by a similarity transformation $A = S^{-1} D S$. Identify S as $\underset{e\leftarrow f}{C}$ or $\underset{f\leftarrow e}{C}$.

Since $\underset{e\leftarrow e}{A} = \underset{e\leftarrow f}{C}\ \underset{f\leftarrow f}{D}\ \underset{f\leftarrow e}{C}$ we identify $S = \underset{f\leftarrow e}{C}$.

h. (4 pts) Compute A^{12} and A^{25}.

With $D = \begin{pmatrix} -6 & 0 \\ 0 & 6 \end{pmatrix} = 6\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ we have

$$D^{12} = \begin{pmatrix} 6^{12} & 0 \\ 0 & 6^{12} \end{pmatrix} = 6^{12}\mathbf{1} \quad \text{and} \quad A^{12} = (S^{-1} D S)^{12} = S^{-1} D^{12} S = 6^{12} S^{-1} \mathbf{1} S = 6^{12}\mathbf{1} = \begin{pmatrix} 6^{12} & 0 \\ 0 & 6^{12} \end{pmatrix}$$

$$D^{25} = 6^{25}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad A^{25} = (S^{-1} D S)^{25} = S^{-1} D^{25} S = 6^{25} S^{-1}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} S$$

$$= 6^{24} S^{-1} D S = 6^{24} A = 6^{24}\begin{pmatrix} 0 & 6 \\ 6 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 6^{25} \\ 6^{25} & 0 \end{pmatrix}$$

3. $\big($30 points$\big)$ The density, ρ, of an ideal gas is related to its pressure, P, and its absolute temperature, T, by the equation $\rho = \frac{P}{kT}$ where k is a constant which depends on the particular ideal gas. We are considering an ideal gas for which $k = 10^{-4}$ atm•m^3/kg/°K. At the current time, $t = t_0$, a flying robotic nanobot is located at $(x,y,z) = (2,1,3)^{\mathsf{T}}$ m and has velocity $\vec{v} = (.4,.5,.2)^{\mathsf{T}}$ m/sec. The nanobot measures the current pressure is $P = 2$ atm while its gradient is $\vec{\nabla}P = (-.03,.01,.02)$atm/m. Similarly, the nanobot measures the current temperature is $T = 250$ °K while its gradient is $\vec{\nabla}T = (3,-2,-4)$ °K/m.

a. $\big($2 pts$\big)$ Find the current density, ρ.

$$\rho = \frac{P}{kT} = \frac{2 \text{ atm}}{\left(10^{-4} \text{ atm} \cdot \text{m}^3/\text{kg}/°\text{K}\right)\left(250\ °\text{K}\right)} = 80 \text{ kg/m}^3$$

b. $\big($6 pts$\big)$ Find the Jacobian matrix of the density $\frac{D(\rho)}{D(P,T)}$ in general (in terms of symbols like $\frac{\partial \rho}{\partial T}$), then in terms of P and T, and finally at the current time $t = t_0$.

$$\frac{D(\rho)}{D(P,T)} = \left(\frac{\partial \rho}{\partial P}, \frac{\partial \rho}{\partial T}\right) = \left(\frac{1}{kT}, \frac{-P}{kT^2}\right) \qquad \frac{D(\rho)}{D(\rho,T)}\bigg|_{t=t_0} = \left(\frac{1}{10^{-4} \cdot 250}, \frac{-2.}{10^{-4}(250)^2}\right) = (40,-.32)$$

c. $\big($4 pts$\big)$ Find the Jacobian matrix $\frac{D(P,T)}{D(x,y,z)}$ in general (in terms of symbols like $\frac{\partial P}{\partial y}$) and then at the current time $t = t_0$.

$$\frac{D(P,T)}{D(x,y,z)} = \begin{pmatrix} \frac{\partial P}{\partial x} & \frac{\partial P}{\partial y} & \frac{\partial P}{\partial z} \\ \frac{\partial T}{\partial x} & \frac{\partial T}{\partial y} & \frac{\partial T}{\partial z} \end{pmatrix} = \begin{pmatrix} \vec{\nabla}P \\ \vec{\nabla}T \end{pmatrix} \qquad \frac{D(\rho,T)}{D(x,y,z)}\bigg|_{t=t_0} = \begin{pmatrix} -.03 & .01 & .02 \\ 3 & -2 & -4 \end{pmatrix}$$

d. $\big($4 pts$\big)$ Find the Jacobian matrix $\frac{D(x,y,z)}{D(t)}$ in general and then at $t = t_0$.

$$\frac{D(x,y,z)}{D(t)} = \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dz}{dt} \end{pmatrix} = \vec{v} \qquad \frac{D(x,y,z)}{D(t)}\bigg|_{t=t_0} = \vec{v}(t_0) = \begin{pmatrix} .4 \\ .5 \\ .2 \end{pmatrix}$$

e. (6 pts) Find the time rate of change of the pressure as seen by the nanobot, at the current time $t = t_0$. Is the pressure currently increasing or decreasing?

$$\left.\frac{dP}{dt}\right|_{t=t_0} = \left.\vec{\nabla}P\right|_{t=t_0} \bullet \vec{v}(t_0) = (-.03,.01,.02) \bullet \begin{pmatrix} .4 \\ .5 \\ .2 \end{pmatrix} = -.012 + .005 + .004 = -.003$$

The pressure is decreasing.

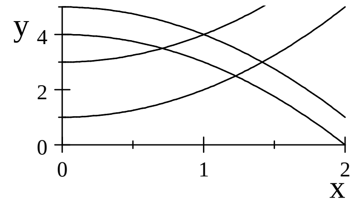
f. (8 pts) Find the time rate of change of the density as seen by the nanobot, at the current time $t = t_0$. Is the density currently increasing or decreasing?

$$\left.\frac{d\rho}{dt}\right|_{t=t_0} = \left.\frac{D(\rho)}{D(P,T)}\right|_{t=t_0} \left.\frac{D(P,T)}{D(x,y,z)}\right|_{t=t_0} \left.\frac{D(x,y,z)}{D(t)}\right|_{t=t_0}$$

$$= (40,-.32)\begin{pmatrix} -.03 & .01 & .02 \\ 3 & -2 & -4 \end{pmatrix}\begin{pmatrix} .4 \\ .5 \\ .2 \end{pmatrix} = (40,-.32)\begin{pmatrix} -.003 \\ -.6 \end{pmatrix} = -.12 + .192 = .072$$

The density is increasing.

4. (26 points) Compute the integral $\iint x\,dA$ over the region in the first quadrant bounded by
$y = 1 + x^2$, $y = 3 + x^2$, $y = 4 - x^2$, and $y = 5 - x^2$.



a. (4 pts) Define the curvilinear coordinates u and v by $y = u + x^2$ and $y = v - x^2$.
What are the 4 boundaries in terms of u and v?

$u = 1 \qquad u = 3 \qquad v = 4 \qquad v = 5$

b. (4 pts) Solve for x and y in terms of u and v. Express the results as a position vector.

Add and subtract: $2y = u + x^2 + v - x^2 = u + v \qquad y = \dfrac{u+v}{2}$

$y - y = u + x^2 - v + x^2 = u - v + 2x^2 \qquad 2x^2 = v - u \qquad x = \dfrac{\sqrt{v-u}}{\sqrt{2}}$

$$\vec{r}(u,v) = (x(u,v), y(u,v)) = \left(\frac{\sqrt{v-u}}{\sqrt{2}}, \frac{u+v}{2} \right)$$

c. (4 pts) Find the coordinate tangent vectors:

$$\vec{e}_u = \frac{\partial \vec{r}}{\partial u} = \left(\frac{1}{2\sqrt{2}} \frac{-1}{\sqrt{v-u}}, \frac{1}{2} \right)$$

$$\vec{e}_v = \frac{\partial \vec{r}}{\partial v} = \left(\frac{1}{2\sqrt{2}} \frac{1}{\sqrt{v-u}}, \frac{1}{2} \right)$$

d. (8 pts) Compute the Jacobian factor:

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{2\sqrt{2}} \frac{-1}{\sqrt{v-u}} & \frac{1}{2} \\ \frac{1}{2\sqrt{2}} \frac{1}{\sqrt{v-u}} & \frac{1}{2} \end{vmatrix} = \frac{1}{4\sqrt{2}} \frac{-1}{\sqrt{v-u}} - \frac{1}{4\sqrt{2}} \frac{1}{\sqrt{v-u}} = \frac{1}{2\sqrt{2}} \frac{-1}{\sqrt{v-u}}$$

$$J = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \frac{1}{2\sqrt{2}} \frac{1}{\sqrt{v-u}}$$

e. (6 pts) Compute the integral:

$$\iint x\,dA = \int_4^5 \int_1^3 \frac{\sqrt{v-u}}{\sqrt{2}} \frac{1}{2\sqrt{2}} \frac{1}{\sqrt{v-u}}\,du\,dv = \int_4^5 \int_1^3 \frac{1}{4}\,du\,dv = \frac{1}{4}(5-4)(3-1) = \frac{1}{2}$$