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Math 311 Final Exam Version A Spring 2015

Section 502 Solutions P. Yasskin

1-8	/40	10	/20
9	/16	11	/28
		Total	/104

Multiple Choice: 5 points each. No Part Credit

Work Out: Points indicated. Show all work.

1. Hydrocloric acid (HCl) and sodium hydroxide $(NaOH)$ react to produce sodium chloride $(NaCl)$ and water (H_2O) according to the chemical equation:

$$aHCl + bNaOH \to cNaCl + dH_2O$$

Which of the following is the augmented matrix which is used to solve this chemical equation? (Put the elements in the order H, Cl, Na, O.)

a. $\left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 & 0 \\ -2 & 0 & 0 & -1 & 0 \end{array}\right)$ **b.** $\left(\begin{array}{cccc|c} 1 & 1 & 0 & 2 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{array}\right)$

c. $\left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 2 & 0 & 0 & 1 & 0 \end{array}\right)$ **d.** $\left(\begin{array}{cccc|c} 1 & 1 & 0 & -2 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \end{array}\right)$ Correct Answer

Solution:
$$\begin{aligned} H: &\quad a + b = 2d \\ Cl: &\quad a = c \\ Na: &\quad b = c \\ O: &\quad b = d \end{aligned}$$
Each equation gets a row.

2. Suppose A is nilpotent, i.e. $A^2 = 0$. Which of the following is true?

a. $A + \mathbf{1}$ is invertible and $(A + \mathbf{1})^{-1} = A - \mathbf{1}$

b. $A + \mathbf{1}$ is invertible and $(A + \mathbf{1})^{-1} = \mathbf{1} - \mathbf{2A}$

c. $A + \mathbf{1}$ is invertible and $(A + \mathbf{1})^{-1} = \mathbf{1} - A$ Correct Answer

d. $A + \mathbf{1}$ is invertible and $(A + \mathbf{1})^{-1} = \mathbf{1} + A$

e. $A + \mathbf{1}$ is not invertible

Solution: $(A + \mathbf{1})(\mathbf{1} - A) = A + \mathbf{1} - \mathbf{A}^2 - A = \mathbf{1} - \mathbf{A}^2 = \mathbf{1}$.

3. Consider the vector space $M(2,2)$ of 2×2 matrices with the basis

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad E_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad E_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad E_4 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Which of the following are the components of the matrix $A = \begin{pmatrix} 9 & 5 \\ 1 & 1 \end{pmatrix}$ relative to the E basis?

a. $(A)_E = \begin{pmatrix} 2 & 1 & 4 & 3 \end{pmatrix}^T$

b. $(A)_E = \begin{pmatrix} 5 & 4 & 3 & 2 \end{pmatrix}^T$ Correct Answer

c. $(A)_E = \begin{pmatrix} 4 & 5 & 2 & 3 \end{pmatrix}^T$

d. $(A)_E = \begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix}^T$

e. $(A)_E = \begin{pmatrix} 5 & -4 & 3 & -2 \end{pmatrix}^T$

$$A = 2E_1 + 1E_2 + 4E_3 + 3E_4 = \begin{pmatrix} 3 & 7 \\ 1 & 1 \end{pmatrix} \qquad A = 5E_1 + 4E_2 + 3E_3 + 2E_4 = \begin{pmatrix} 9 & 5 \\ 1 & 1 \end{pmatrix}$$

$$A = 4E_1 + 5E_2 + 2E_3 + 3E_4 = \begin{pmatrix} 9 & 5 \\ -1 & -1 \end{pmatrix} \qquad A = 1E_1 + 2E_2 + 3E_3 + 4E_4 = \begin{pmatrix} 3 & 7 \\ -1 & -1 \end{pmatrix}$$

$$A = 5E_1 - 4E_2 + 3E_3 - 2E_4 = \begin{pmatrix} 1 & 1 \\ 5 & 9 \end{pmatrix}$$

4. Consider the vector space $C_2([0,1]$ of real valued function on the interval $[0,1]$ whose second derivatives exist and are continuous with the inner product $\langle f,g \rangle = \int_{-1}^{1} f(x)\,g(x)\,dx$. Consider the subspace $V = Span(x, x^2)$ spanned by the basis $v_1 = x$, $v_2 = x^2$. Which of the following is an orthonormal basis for V?

a. $u_1 = x \qquad u_2 = x^2$

b. $u_1 = \sqrt{\frac{5}{2}}\,x \qquad u_2 = \sqrt{\frac{7}{2}}\,x^2$

c. $u_1 = x \qquad u_2 = x^2 - \frac{3}{4}x$

d. $u_1 = \sqrt{\frac{3}{2}}\,x \qquad u_2 = \sqrt{\frac{5}{2}}\,x^2$ Correct Answer

e. $u_1 = \sqrt{2}\,x \qquad u_2 = 4\sqrt{5}\,x^2 - 3\sqrt{5}\,x$

We apply Gram-Schmidt to $v_1 = x$, $v_2 = x^2$

$$w_1 = x \qquad \langle w_1, w_1 \rangle = \langle x, x \rangle = \int_{-1}^{1} x^2\,dx = \frac{2}{3} \qquad u_1 = \sqrt{\frac{3}{2}}\,x$$

$$\langle v_2, w_1 \rangle = \langle x^2, x \rangle = \int_{-1}^{1} x^3\,dx = 0 \qquad w_2 = v_2 = x^2$$

$$\langle w_2, w_2 \rangle = \langle x^2, x^2 \rangle = \int_{-1}^{1} x^4\,dx = \frac{2}{5} \qquad u_2 = \sqrt{\frac{5}{2}}\,x^2$$

5. Consider the second derivative linear operator $L : P_6 \to P_6 : L(p) = \dfrac{d^2p}{dx^2}$ on the space of polynomials of degree less than 6. Find the image, $\text{Im}(L)$.
HINT: Let $p = a + bx + cx^2 + dx^3 + ex^4 + fx^5$.

a. $\text{Im}(L) = Span(1)$
b. $\text{Im}(L) = Span(1,x)$
c. $\text{Im}(L) = Span(1,x,x^2,x^3)$ Correct Answer
d. $\text{Im}(L) = Span(x^2,x^3,x^4,x^5)$
e. $\text{Im}(L) = Span(x,x^2,x^3,x^4,x^5)$

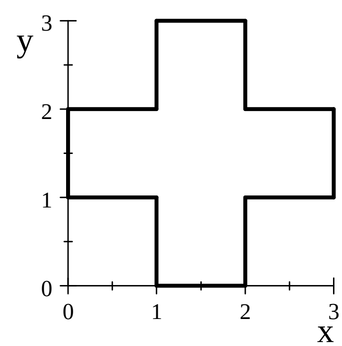
$\text{Im}(L) = \{L(p)\} = \{2c + 6dx + 12ex^2 + 20fx^3\} = Span(1,x,x^2,x^3)$

6. Consider the second derivative linear operator $L : P_6 \to P_6 : L(p) = \dfrac{d^2p}{dx^2}$ on the space of polynomials of degree less than 6. Find the kernel, $Ker(L)$.
HINT: Let $p = a + bx + cx^2 + dx^3 + ex^4 + fx^5$.

a. $Ker(L) = Span(1)$
b. $Ker(L) = Span(1,x)$ Correct Answer
c. $Ker(L) = Span(1,x,x^2)$
d. $Ker(L) = Span(x^2,x^3,x^4,x^5)$
e. $Ker(L) = Span(x,x^2,x^3,x^4,x^5)$

$Ker(L) = \{p \mid L(p) = 0\}$
$L(p) = 2c + 6dx + 12ex^2 + 20fx^3 = 0 \quad \Rightarrow \quad c = d = e = f = 0$
$Ker(L) = \{p = a + bx\} = Span(1,x)$

7. Compute the line integral $\oint \vec{F} \cdot d\vec{s}$ clockwise around the complete boundary of the plus sign, shown at the right, for the vector field $\vec{F} = (4x^3 + 2y, 4y^3 - 3x)$.



a. -25
b. -5
c. 0
d. 5
e. 25 Correct Answer

We let $(P,Q) = \vec{F} = (4x^3 + 2y, 4y^3 - 3x)$ and apply Green's Theorem.
There is an extra minus sign because we need $\oint \vec{F} \cdot d\vec{s}$ clockwise, but Green's Theorem expects counterclockwise:

$$\oint \vec{F} \cdot d\vec{s} = \oint P\,dx + Q\,dy = -\iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy = -\iint (-3-2)\,dx\,dy = 5 \cdot Area = 5 \cdot 5 = 25$$

8. Consider the vector space $M(2,2)$ of 2×2 matrices. Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. Consider the linear function, $L : M(2,2) \to M(2,2) : L(X) = AX - XA$. Which of the following is not an eigenvalue and corresponding eigenmatrix (eigenvector) of L?

HINT: Let $X = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$.

Solution:

$$L(X) = L\begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x & y \\ z & w \end{pmatrix} - \begin{pmatrix} x & y \\ z & w \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & -y \\ z & 0 \end{pmatrix}$$

a. $\lambda = 2$ $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ Correct Answer $L\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \neq 2\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

b. $\lambda = 1$ $X = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $L\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = 1\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

c. $\lambda = 0$ $X = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $L\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

d. $\lambda = 0$ $X = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ $L\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

e. $\lambda = -1$ $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $L\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} = -1\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

9. (16 points) Which of the following is an inner product on $\mathbb{R}^2$? If not, why not?
Put $\times$'s in the correct boxes. No part credit.
Let $\vec{x} = (x_1, x_2)$, $\vec{y} = (y_1, y_2)$.

		Inner Product?		Why not?			
				Not	Not	Not	Positive but Not
	$\langle \vec{x}, \vec{y} \rangle =$	Yes	No	Symmetric	Linear	Positive	Positive Definite
a.	$x_1y_1 + 2x_2y_2$	×					
b.	$x_1y_1 + 2x_1y_2 + 2x_2y_2$		×	×			
c.	$x_1^2y_1^2 + 2x_2^2y_2^2$		×		×		
d.	$x_1y_1 + x_1y_2 + x_2y_1 + 2x_2y_2$	×					
e.	$x_1y_1 - x_1y_2 - x_2y_1 + 2x_2y_2$	×					
f.	$x_1y_1 - x_1y_2 - x_2y_1 + x_2y_2$		×				×
g.	$x_1y_1 + x_1y_2 + x_2y_1 + x_2y_2$		×				×
h.	$x_1y_1 - x_2y_2$		×			×	

Solutions:

a. $\langle\vec{y},\vec{x}\rangle = y_1x_1 + 2y_2x_2 = \langle\vec{x},\vec{y}\rangle \qquad \langle\vec{x},a\vec{y}+b\vec{z}\rangle = a\langle\vec{x},\vec{y}\rangle + b\langle\vec{x},\vec{z}\rangle$

$\langle\vec{x},\vec{x}\rangle = x_1^2 + 2x_2^2 \geq 0$ and $= 0$ only if $x_1 = x_2 = 0$

b. $\langle\vec{y},\vec{x}\rangle = y_1x_1 + 2y_1x_2 + 2y_2x_2 \neq \langle\vec{x},\vec{y}\rangle$

c. $\langle\vec{x},a\vec{y}\rangle = a^2x_1^2y_1^2 + 2a^2x_2^2y_2^2 = a^2\langle\vec{x},\vec{y}\rangle \neq a\langle\vec{x},\vec{y}\rangle$

d. $\langle\vec{y},\vec{x}\rangle = y_1x_1 + y_1x_2 + y_2x_1 + 2y_2x_2 = \langle\vec{x},\vec{y}\rangle \qquad \langle\vec{x},a\vec{y}+b\vec{z}\rangle = a\langle\vec{x},\vec{y}\rangle + b\langle\vec{x},\vec{z}\rangle$

$\langle\vec{x},\vec{x}\rangle = x_1^2 + 2x_1x_2 + 2x_2^2 = (x_1+x_2)^2 + x_2^2 \geq 0$ and $= 0$ only if $x_1 = x_2 = 0$

e. $\langle\vec{y},\vec{x}\rangle = y_1x_1 - y_1x_2 - y_2x_1 + 2y_2x_2 = \langle\vec{x},\vec{y}\rangle \qquad \langle\vec{x},a\vec{y}+b\vec{z}\rangle = a\langle\vec{x},\vec{y}\rangle + b\langle\vec{x},\vec{z}\rangle$

$\langle\vec{x},\vec{x}\rangle = x_1^2 - 2x_1x_2 + 2x_2^2 = (x_1-x_2)^2 + x_2^2 \geq 0$ and $= 0$ only if $x_1 = x_2 = 0$

f. $\langle\vec{x},\vec{x}\rangle = x_1^2 - 2x_1x_2 + x_2^2 = (x_1-x_2)^2 \geq 0$ but can $= 0$ if $x_1 = x_2 \neq 0$

g. $\langle\vec{x},\vec{x}\rangle = x_1^2 + 2x_1x_2 + x_2^2 = (x_1+x_2)^2 \geq 0$ but can $= 0$ if $x_1 = -x_2 \neq 0$

h. $\langle\vec{x},\vec{x}\rangle = x_1^2 - x_2^2 < 0$ if $x_2 > x_1 > 0$

10. (20 points) Let $M(2,3)$ be the vector space of 2×3 matrices.
Consider the subspace $V = Span(A_1,A_2,A_3,A_4)$ where

$$A_1 = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 3 & -2 \end{pmatrix},\quad A_2 = \begin{pmatrix} 2 & 2 & 0 \\ 4 & 6 & -4 \end{pmatrix},\quad A_3 = \begin{pmatrix} -1 & 0 & 2 \\ -3 & 0 & 0 \end{pmatrix},\quad A_4 = \begin{pmatrix} -1 & 2 & 6 \\ -5 & 6 & -4 \end{pmatrix}$$

Find a basis for V. What is the $\dim V$?

Solution: To see if A_1, A_2, A_3, A_4 are linearly independent, we write:
$aA_1 + bA_2 + cA_3 + dA_4 = \mathbf{0}$ and solve for a, b, c, d.

$$a\begin{pmatrix} 1 & 1 & 0 \\ 2 & 3 & -2 \end{pmatrix} + b\begin{pmatrix} 2 & 2 & 0 \\ 4 & 6 & -4 \end{pmatrix} + c\begin{pmatrix} -1 & 0 & 2 \\ -3 & 0 & 0 \end{pmatrix} + d\begin{pmatrix} -1 & 2 & 6 \\ -5 & 6 & -4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} a+2b-c-d & a+2b+2d & 2c+6d \\ 2a+4b-3c-5d & 3a+6b+6d & -2a-4b-4d \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{matrix}\text{6 equations} \\ \text{4 unknowns}\end{matrix}$$

$$\left(\begin{array}{cccc|c} 1 & 2 & -1 & -1 & 0 \\ 1 & 2 & 0 & 2 & 0 \\ 0 & 0 & 2 & 6 & 0 \\ 2 & 4 & -3 & -5 & 0 \\ 3 & 6 & 0 & 6 & 0 \\ -2 & -4 & 0 & -4 & 0 \end{array}\right) \begin{matrix} \\ R_2 - R_1 \\ \\ R_4 - 2R_1 \\ R_5 - 3R_1 \\ R_6 + 2R_1 \end{matrix} \Rightarrow \left(\begin{array}{cccc|c} 1 & 2 & -1 & -1 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 2 & 6 & 0 \\ 0 & 0 & -1 & -3 & 0 \\ 0 & 0 & 3 & 9 & 0 \\ 0 & 0 & -2 & -6 & 0 \end{array}\right) \begin{matrix} R_1 + R_2 \\ \\ R_3 - 2R_2 \\ R_4 + R_2 \\ R_5 - 3R_2 \\ R_6 + 2R_2 \end{matrix} \Rightarrow \left(\begin{array}{cccc|c} 1 & 2 & 0 & 2 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$\Rightarrow \quad a + 2b + 2d = 0, \quad c + 3d = 0 \quad \Rightarrow \quad a = -2r - 2s \quad b = r \quad c = -3s \quad d = s$

$(-2r-2s)A_1 + rA_2 - 3sA_3 + sA_4 = 0$ So they are not linearly independent.

Further, with $r = 1$ and $s = 0$: $A_2 = 2A_1$ and with $r = 0$ and $s = 1$: $A_4 = 2A_1 + 3A_3$

So A_1, A_2 also span and the above computation without columns 2 and 4 shows they are linearly independent. So A_1, A_2 is a basis and $\dim V = 2$.

11. (28 points) Compute $\iint\limits_H \vec{F} \cdot d\vec{S}$ over the hemisphere $z = \sqrt{25 - x^2 - y^2}$ oriented upward, for the vector field $\vec{F} = (x^3 - 4y^2 - 4z^2, -4x^2 + y^3 - 4z^2, -4x^2 - 4y^2 + z^3)$.

HINT: Use Gauss' Theorem by following these steps:

a. Write out Gauss' Theorem for the Volume, V, which is the solid hemisphere $0 \le z \le \sqrt{25 - x^2 - y^2}$. Split up the boundary, ∂V, into two pieces, the hemisphere, H, and the disk, D, at the bottom. State orientations. Solve for the integral you want.

b. Compute the volume integral using spherical coordinates.

c. Compute the other surface integral over D by parametrizing the disk, computing the tangent vectors and normal vector, checking the orientation, evaluating the vector field on the disk and doing the integral.

d. Solve for the original integral.

Solution:

a. Gauss' Theorem: $\iiint\limits_V \vec{\nabla} \cdot \vec{F}\, dV = \iint\limits_{\partial V} \vec{F} \cdot d\vec{S} = \iint\limits_H \vec{F} \cdot d\vec{S} + \iint\limits_D \vec{F} \cdot d\vec{S}$

H is oriented upward. D is oriented downward.

$$\iint\limits_H \vec{F} \cdot d\vec{S} = \iiint\limits_V \vec{\nabla} \cdot \vec{F}\, dV - \iint\limits_D \vec{F} \cdot d\vec{S}$$

b. In spherical coordinates: $dV = \rho^2 \sin\varphi$ and $\vec{\nabla} \cdot \vec{F} = 3x^2 + 3y^2 + 3z^2 = 3\rho^2$.

$$\iiint\limits_V \vec{\nabla} \cdot \vec{F}\, dV = \int_0^{2\pi} \int_0^{\pi/2} \int_0^5 3\rho^2 \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta = 3 \cdot 2\pi \Big[-\cos\theta\Big]_0^{\pi/2} \left[\frac{\rho^5}{5}\right]_0^5 = 6\pi(1)5^4 = 3750\pi$$

c. The disk may be parametrized by $\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, 0)$.

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \vec{e}_r = \cos\theta & \sin\theta & 0 \\ \vec{e}_\theta = -r\sin\theta & r\cos\theta & 0 \end{vmatrix}$$

$\vec{N} = \vec{e}_r \times \vec{e}_\theta = \hat{\imath}(0) - \hat{\jmath}(0) + \hat{k}(r\cos^2\theta + r\sin^2\theta) = (0,0,r)$

Reverse to point down: $\vec{N} = (0,0,-r)$

$\vec{F} = (x^3 - 4y^2 - 4z^2, -4x^2 + y^3 - 4z^2, -4x^2 - 4y^2 + z^3) = (*,*,-4r^2 + 0^3)$ $\quad \vec{F} \cdot \vec{N} = 4r^3$

$$\iint\limits_D \vec{F} \cdot d\vec{S} = \int_0^{2\pi} \int_0^5 \vec{F} \cdot \vec{N}\, dr\, d\theta = \int_0^{2\pi} \int_0^5 4r^3\, dr\, d\theta = 2\pi\Big[r^4\Big]_0^5 = 1250\pi$$

d. $\iint\limits_H \vec{F} \cdot d\vec{S} = \iiint\limits_V \vec{\nabla} \cdot \vec{F}\, dV - \iint\limits_D \vec{F} \cdot d\vec{S} = 3750\pi - 1250\pi = 2500\pi$