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Math 311 Final Exam Version B Spring 2015

Section 502 Solutions P. Yasskin

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|---|---|---|---|
| 1-8 | /40 | 10 | /20 |
| 9 | /16 | 11 | /28 |
| | | Total | /104 |

Multiple Choice: 5 points each. No Part Credit

Work Out: Points indicated. Show all work.

**1**. Hydrocloric acid $(HCl)$ and sodium hydroxide $(NaOH)$ react to produce sodium chloride $(NaCl)$ and water $(H_2O)$ according to the chemical equation:

$$aHCl + bNaOH \to cNaCl + dH_2O$$

Which of the following is the augmented matrix which is used to solve this chemical equation? (Put the elements in the order $H$, $Cl$, $Na$, $O$.)

**a**. $\left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 & 0 \\ -2 & 0 & 0 & -1 & 0 \end{array}\right)$ **b**. $\left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 2 & 0 & 0 & 1 & 0 \end{array}\right)$

**c**. $\left(\begin{array}{cccc|c} 1 & 1 & 0 & -2 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \end{array}\right)$ Correct Answer **d**. $\left(\begin{array}{cccc|c} 1 & 1 & 0 & 2 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{array}\right)$

Solution:
$$\begin{array}{lrl} H: & a+b &= 2d \\ Cl: & a &= c \\ Na: & b &= c \\ O: & b &= d \end{array}$$
Each equation gets a row.

**2**. Suppose $A$ is nilpotent, i.e. $A^2 = 0$. Which of the following is true?

**a**. $A + \mathbf{1}$ is invertible and $(A+\mathbf{1})^{-1} = A - \mathbf{1}$

**b**. $A + \mathbf{1}$ is invertible and $(A+\mathbf{1})^{-1} = \mathbf{1} - A$ Correct Answer

**c**. $A + \mathbf{1}$ is invertible and $(A+\mathbf{1})^{-1} = \mathbf{1} - \mathbf{2A}$

**d**. $A + \mathbf{1}$ is invertible and $(A+\mathbf{1})^{-1} = \mathbf{1} + A$

**e**. $A + \mathbf{1}$ is not invertible

Solution: $(A+\mathbf{1})(\mathbf{1}-A) = A + \mathbf{1} - \mathbf{A}^2 - A = \mathbf{1} - \mathbf{A}^2 = \mathbf{1}$.

**3.** Consider the vector space $M(2,2)$ of $2 \times 2$ matrices with the basis

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad E_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad E_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad E_4 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Which of the following are the components of the matrix $A = \begin{pmatrix} 9 & 5 \\ 1 & 1 \end{pmatrix}$ relative to the $E$ basis?

**a.** $(A)_E = \begin{pmatrix} 2 & 1 & 4 & 3 \end{pmatrix}^T$

**b.** $(A)_E = \begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix}^T$

**c.** $(A)_E = \begin{pmatrix} 4 & 5 & 2 & 3 \end{pmatrix}^T$

**d.** $(A)_E = \begin{pmatrix} 5 & 4 & 3 & 2 \end{pmatrix}^T$ Correct Answer

**e.** $(A)_E = \begin{pmatrix} 5 & -4 & 3 & -2 \end{pmatrix}^T$

$$A = 2E_1 + 1E_2 + 4E_3 + 3E_4 = \begin{pmatrix} 3 & 7 \\ 1 & 1 \end{pmatrix} \qquad A = 1E_1 + 2E_2 + 3E_3 + 4E_4 = \begin{pmatrix} 3 & 7 \\ -1 & -1 \end{pmatrix}$$

$$A = 4E_1 + 5E_2 + 2E_3 + 3E_4 = \begin{pmatrix} 9 & 5 \\ -1 & -1 \end{pmatrix} \qquad A = 5E_1 + 4E_2 + 3E_3 + 2E_4 = \begin{pmatrix} 9 & 5 \\ 1 & 1 \end{pmatrix}$$

$$A = 5E_1 - 4E_2 + 3E_3 - 2E_4 = \begin{pmatrix} 1 & 1 \\ 5 & 9 \end{pmatrix}$$

**4.** Consider the vector space $C_2([0,1]$ of real valued function on the interval $[0,1]$ whose second derivatives exist and are continuous with the inner product $\langle f,g \rangle = \int_{-1}^{1} f(x)\,g(x)\,dx$. Consider the subspace $V = Span(x, x^2)$ spanned by the basis $v_1 = x$, $v_2 = x^2$. Which of the following is an orthonormal basis for $V$?

**a.** $u_1 = x \qquad u_2 = x^2$

**b.** $u_1 = x \qquad u_2 = x^2 - \frac{3}{4}x$

**c.** $u_1 = \sqrt{\frac{3}{2}}\,x \qquad u_2 = \sqrt{\frac{5}{2}}\,x^2$ Correct Answer

**d.** $u_1 = \sqrt{\frac{5}{2}}\,x \qquad u_2 = \sqrt{\frac{7}{2}}\,x^2$

**e.** $u_1 = \sqrt{2}\,x \qquad u_2 = 4\sqrt{5}\,x^2 - 3\sqrt{5}\,x$

We apply Gram-Schmidt to $v_1 = x$, $v_2 = x^2$

$$w_1 = x \qquad \langle w_1, w_1 \rangle = \langle x, x \rangle = \int_{-1}^{1} x^2\,dx = \frac{2}{3} \qquad u_1 = \sqrt{\frac{3}{2}}\,x$$

$$\langle v_2, w_1 \rangle = \langle x^2, x \rangle = \int_{-1}^{1} x^3\,dx = 0 \qquad w_2 = v_2 = x^2$$

$$\langle w_2, w_2 \rangle = \langle x^2, x^2 \rangle = \int_{-1}^{1} x^4\,dx = \frac{2}{5} \qquad u_2 = \sqrt{\frac{5}{2}}\,x^2$$

**5**. Consider the second derivative linear operator $L : P_6 \to P_6 : L(p) = \dfrac{d^2p}{dx^2}$ on the space of polynomials of degree less than $6$. Find the kernel, $Ker(L)$.
HINT: Let $p = a + bx + cx^2 + dx^3 + ex^4 + fx^5$.

**a**. $Ker(L) = Span(1)$
**b**. $Ker(L) = Span(1,x)$ Correct Answer
**c**. $Ker(L) = Span(1,x,x^2)$
**d**. $Ker(L) = Span(x^2,x^3,x^4,x^5)$
**e**. $Ker(L) = Span(x,x^2,x^3,x^4,x^5)$

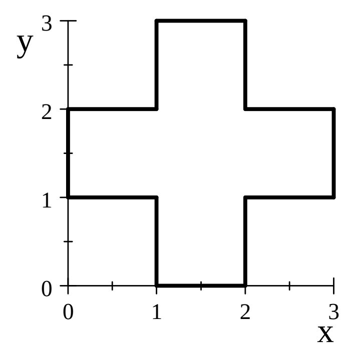
$Ker(L) = \{p \mid L(p) = 0\}$
$L(p) = 2c + 6dx + 12ex^2 + 20fx^3 = 0 \quad \Rightarrow \quad c = d = e = f = 0$
$Ker(L) = \{p = a + bx\} = Span(1,x)$

**6**. Consider the second derivative linear operator $L : P_6 \to P_6 : L(p) = \dfrac{d^2p}{dx^2}$ on the space of polynomials of degree less than $6$. Find the image, $\mathrm{Im}(L)$.
HINT: Let $p = a + bx + cx^2 + dx^3 + ex^4 + fx^5$.

**a**. $\mathrm{Im}(L) = Span(1)$
**b**. $\mathrm{Im}(L) = Span(1,x)$
**c**. $\mathrm{Im}(L) = Span(1,x,x^2,x^3)$ Correct Answer
**d**. $\mathrm{Im}(L) = Span(x^2,x^3,x^4,x^5)$
**e**. $\mathrm{Im}(L) = Span(x,x^2,x^3,x^4,x^5)$

$\mathrm{Im}(L) = \{L(p)\} = \{2c + 6dx + 12ex^2 + 20fx^3\} = Span(1,x,x^2,x^3)$

**7**. Compute the line integral $\oint \vec{F} \cdot d\vec{s}$ clockwise around the complete boundary of the plus sign, shown at the right, for the vector field $\vec{F} = (4x^3 + 2y, 4y^3 - 3x)$.




**a**. 25 Correct Answer
**b**. 5
**c**. 0
**d**. –5
**e**. –25

We let $(P,Q) = \vec{F} = (4x^3 + 2y, 4y^3 - 3x)$ and apply Green's Theorem.
There is an extra minus sign because we need $\oint \vec{F} \cdot d\vec{s}$ clockwise, but Green's Theorem expects counterclockwise:

$$\oint \vec{F} \cdot d\vec{s} = \oint P\,dx + Q\,dy = -\iint \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy = -\iint (-3 - 2)\,dx\,dy = 5 \cdot Area = 5 \cdot 5 = 25$$

**8**. Consider the vector space $M(2,2)$ of $2\times 2$ matrices. Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. Consider the linear function, $L : M(2,2) \to M(2,2) : L(X) = AX - XA$. Which of the following is not an eigenvalue and corresponding eigenmatrix (eigenvector) of $L$?

HINT: Let $X = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$.

Solution:

$$L(X) = L\begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x & y \\ z & w \end{pmatrix} - \begin{pmatrix} x & y \\ z & w \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & -y \\ z & 0 \end{pmatrix}$$

**a**. $\lambda = -1 \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\qquad L\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} = -1\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

**b**. $\lambda = 0 \quad X = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $\qquad L\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

**c**. $\lambda = 0 \quad X = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ $\qquad L\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

**d**. $\lambda = 1 \quad X = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $\qquad L\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = 1\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

**e**. $\lambda = 2 \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ Correct Answer $\qquad L\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \neq 2\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

**9**. (16 points) Which of the following is an inner product on $\mathbb{R}^2$? If not, why not?
Put $\times$'s in the correct boxes. No part credit.
Let $\vec{x} = (x_1, x_2)$, $\vec{y} = (y_1, y_2)$.

| | | Inner Product? | | Why not? | | | |
|---|---|---|---|---|---|---|---|
| | | | | Not | Not | Not | Positive but Not |
| | $\langle\vec{x},\vec{y}\rangle =$ | Yes | No | Symmetric | Linear | Positive | Positive Definite |
| **a**. | $x_1y_1 + 2x_2y_2$ | × | | | | | |
| **b**. | $x_1^2y_1^2 + 2x_2^2y_2^2$ | | × | | × | | |
| **c**. | $x_1y_1 + 2x_1y_2 + 2x_2y_2$ | | × | × | | | |
| **d**. | $x_1y_1 - x_1y_2 - x_2y_1 + x_2y_2$ | | × | | | | × |
| **e**. | $x_1y_1 + x_1y_2 + x_2y_1 + x_2y_2$ | | × | | | | × |
| **f**. | $x_1y_1 + x_1y_2 + x_2y_1 + 2x_2y_2$ | × | | | | | |
| **g**. | $x_1y_1 - x_1y_2 - x_2y_1 + 2x_2y_2$ | × | | | | | |
| **h**. | $x_1y_1 - x_2y_2$ | | × | | | × | |

Solutions:

**a.** $\langle \vec{y},\vec{x}\rangle = y_1x_1 + 2y_2x_2 = \langle \vec{x},\vec{y}\rangle \qquad \langle \vec{x}, a\vec{y} + b\vec{z}\rangle = a\langle \vec{x},\vec{y}\rangle + b\langle \vec{x},\vec{z}\rangle$

$\langle \vec{x},\vec{x}\rangle = x_1^2 + 2x_2^2 \geq 0$ and $= 0$ only if $x_1 = x_2 = 0$

**b.** $\langle \vec{x}, a\vec{y}\rangle = a^2x_1^2y_1^2 + 2a^2x_2^2y_2^2 = a^2\langle \vec{x},\vec{y}\rangle \neq a\langle \vec{x},\vec{y}\rangle$

**c.** $\langle \vec{y},\vec{x}\rangle = y_1x_1 + 2y_1x_2 + 2y_2x_2 \neq \langle \vec{x},\vec{y}\rangle$

**d.** $\langle \vec{x},\vec{x}\rangle = x_1^2 - 2x_1x_2 + x_2^2 = (x_1 - x_2)^2 \geq 0$ but can $= 0$ if $x_1 = x_2 \neq 0$

**e.** $\langle \vec{x},\vec{x}\rangle = x_1^2 + 2x_1x_2 + x_2^2 = (x_1 + x_2)^2 \geq 0$ but can $= 0$ if $x_1 = -x_2 \neq 0$

**f.** $\langle \vec{y},\vec{x}\rangle = y_1x_1 + y_1x_2 + y_2x_1 + 2y_2x_2 = \langle \vec{x},\vec{y}\rangle \qquad \langle \vec{x}, a\vec{y} + b\vec{z}\rangle = a\langle \vec{x},\vec{y}\rangle + b\langle \vec{x},\vec{z}\rangle$

$\langle \vec{x},\vec{x}\rangle = x_1^2 + 2x_1x_2 + 2x_2^2 = (x_1 + x_2)^2 + x_2^2 \geq 0$ and $= 0$ only if $x_1 = x_2 = 0$

**g.** $\langle \vec{y},\vec{x}\rangle = y_1x_1 - y_1x_2 - y_2x_1 + 2y_2x_2 = \langle \vec{x},\vec{y}\rangle \qquad \langle \vec{x}, a\vec{y} + b\vec{z}\rangle = a\langle \vec{x},\vec{y}\rangle + b\langle \vec{x},\vec{z}\rangle$

$\langle \vec{x},\vec{x}\rangle = x_1^2 - 2x_1x_2 + 2x_2^2 = (x_1 - x_2)^2 + x_2^2 \geq 0$ and $= 0$ only if $x_1 = x_2 = 0$

**h.** $\langle \vec{x},\vec{x}\rangle = x_1^2 - x_2^2 < 0$ if $x_2 > x_1 > 0$

**10.** (20 points) Let $M(2,3)$ be the vector space of $2 \times 3$ matrices.

Consider the subspace $V = Span(A_1, A_2, A_3, A_4)$ where

$$A_1 = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 3 & -2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 2 & 2 & 0 \\ 4 & 6 & -4 \end{pmatrix}, \quad A_3 = \begin{pmatrix} -1 & 0 & 2 \\ -3 & 0 & 0 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 1 & 3 & 4 \\ 0 & 9 & -6 \end{pmatrix}$$

Find a basis for $V$. What is the $\dim V$ ?

Solution: To see if $A_1$, $A_2$, $A_3$, $A_4$ are linearly independent, we write:

$aA_1 + bA_2 + cA_3 + dA_4 = \mathbf{0}$ and solve for $a$, $b$, $c$, $d$.

$$a\begin{pmatrix} 1 & 1 & 0 \\ 2 & 3 & -2 \end{pmatrix} + b\begin{pmatrix} 2 & 2 & 0 \\ 4 & 6 & -4 \end{pmatrix} + c\begin{pmatrix} -1 & 0 & 2 \\ -3 & 0 & 0 \end{pmatrix} + d\begin{pmatrix} 1 & 3 & 4 \\ 0 & 9 & -6 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} a + 2b - c + d & a + 2b + 3d & 2c + 4d \\ 2a + 4b - 3c & 3a + 6b + 9d & -2a - 4b - 6d \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

6 equations
4 unknowns

$$\left(\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 0 \\ 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 2 & 4 & 0 \\ 2 & 4 & -3 & 0 & 0 \\ 3 & 6 & 0 & 9 & 0 \\ -2 & -4 & 0 & -6 & 0 \end{array}\right) \begin{array}{c} \\ R_2 - R_1 \\ \\ R_4 - 2R_1 \\ R_5 - 3R_1 \\ R_6 + 2R_1 \end{array} \Rightarrow \left(\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 4 & 0 \\ 0 & 0 & -1 & -2 & 0 \\ 0 & 0 & 3 & 6 & 0 \\ 0 & 0 & -2 & -4 & 0 \end{array}\right) \begin{array}{c} R_1 + R_2 \\ \\ R_3 - 2R_2 \\ R_4 + R_2 \\ R_5 - 3R_2 \\ R_6 + 2R_2 \end{array} \Rightarrow \left(\begin{array}{cccc|c} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$\Rightarrow \quad a + 2b + 3d = 0, \quad c + 2d = 0 \quad \Rightarrow \quad a = -2r - 3s \quad b = r \quad c = -2s \quad d = s$

$(-2r - 3s)A_1 + rA_2 - 2sA_3 + sA_4 = 0$ So they are not linearly independent.

Further, with $r = 1$ and $s = 0$: $A_2 = 2A_1$ and with $r = 0$ and $s = 1$: $A_4 = 3A_1 + 2A_3$

So $A_1, A_2$ also span and the above computation without columns 2 and 4 shows they are linearly independent. So $A_1, A_2$ is a basis and $\dim V = 2$.

**11.** (28 points) Compute $\iint\limits_H \vec{F}\cdot d\vec{S}$ over the hemisphere $z=\sqrt{25-x^2-y^2}$ oriented upward, for the vector field $\vec{F}=(x^3+4y^2+4z^2, 4x^2+y^3+4z^2, 4x^2+4y^2+z^3)$.

HINT: Use Gauss' Theorem by following these steps:

**a.** Write out Gauss' Theorem for the Volume, $V$, which is the solid hemisphere $0\le z\le\sqrt{25-x^2-y^2}$. Split up the boundary, $\partial V$, into two pieces, the hemisphere, $H$, and the disk, $D$, at the bottom. State orientations. Solve for the integral you want.

**b.** Compute the volume integral using spherical coordinates.

**c.** Compute the other surface integral over $D$ by parametrizing the disk, computing the tangent vectors and normal vector, checking the orientation, evaluating the vector field on the disk and doing the integral.

**d.** Solve for the original integral.

Solution:

**a.** Gauss' Theorem: $\iiint\limits_V \vec{\nabla}\cdot\vec{F}\,dV=\iint\limits_{\partial V}\vec{F}\cdot d\vec{S}=\iint\limits_H \vec{F}\cdot d\vec{S}+\iint\limits_D \vec{F}\cdot d\vec{S}$

$H$ is oriented upward. $D$ is oriented downward.

$$\iint\limits_H \vec{F}\cdot d\vec{S}=\iiint\limits_V \vec{\nabla}\cdot\vec{F}\,dV-\iint\limits_D \vec{F}\cdot d\vec{S}$$

**b.** In spherical coordinates: $dV=\rho^2\sin\varphi$ and $\vec{\nabla}\cdot\vec{F}=3x^2+3y^2+3z^2=3\rho^2$.

$$\iiint\limits_V \vec{\nabla}\cdot\vec{F}\,dV=\int_0^{2\pi}\int_0^{\pi/2}\int_0^5 3\rho^2\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta=3\cdot 2\pi\Big[-\cos\theta\Big]_0^{\pi/2}\left[\frac{\rho^5}{5}\right]_0^5=6\pi(1)5^4=3750\pi$$

**c.** The disk may be parametrized by $\vec{R}(r,\theta)=(r\cos\theta, r\sin\theta, 0)$.

$$\begin{array}{r|ccc|} & \hat{\imath} & \hat{\jmath} & \hat{k}\\ \vec{e}_r= & \cos\theta & \sin\theta & 0\\ \vec{e}_\theta= & -r\sin\theta & r\cos\theta & 0\end{array}$$

$\vec{N}=\vec{e}_r\times\vec{e}_\theta=\hat{\imath}(0)-\hat{\jmath}(0)+\hat{k}(r\cos^2\theta+r\sin^2\theta)=(0,0,r)$

Reverse to point down: $\vec{N}=(0,0,-r)$

$\vec{F}=(x^3+4y^2+4z^2, 4x^2+y^3+4z^2, 4x^2+4y^2+z^3)=(*,*,4r^2+0^3)$ $\quad\vec{F}\cdot\vec{N}=-4r^3$

$$\iint\limits_D \vec{F}\cdot d\vec{S}=\int_0^{2\pi}\int_0^5 \vec{F}\cdot\vec{N}\,dr\,d\theta=-\int_0^{2\pi}\int_0^5 4r^3\,dr\,d\theta=-2\pi\Big[r^4\Big]_0^5=-1250\pi$$

**d.** $\iint\limits_H \vec{F}\cdot d\vec{S}=\iiint\limits_V \vec{\nabla}\cdot\vec{F}\,dV-\iint\limits_D \vec{F}\cdot d\vec{S}=3750\pi--1250\pi=5000\pi$