

Name \_\_\_\_\_

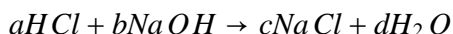
Math 311      Final Exam Version A      Spring 2015  
 Section 503      Solutions      P. Yasskin

1-8	/40	10	/20
9	/16	11	/28
		Total	/104

Multiple Choice: 5 points each. No Part Credit

Work Out: Points indicated. Show all work.

1. Hydrochloric acid ( $HCl$ ) and sodium hydroxide ( $NaOH$ ) react to produce sodium chloride ( $NaCl$ ) and water ( $H_2O$ ) according to the chemical equation:



Which of the following is the augmented matrix which is used to solve this chemical equation?  
 (Put the elements in the order  $H, Cl, Na, O$ .)

a.  $\left( \begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 & 0 \\ -2 & 0 & 0 & -1 & 0 \end{array} \right)$

b.  $\left( \begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 2 & 0 & 0 & 1 & 0 \end{array} \right)$

c.  $\left( \begin{array}{cccc|c} 1 & 1 & 0 & -2 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \end{array} \right)$

Correct Answer

d.  $\left( \begin{array}{cccc|c} 1 & 1 & 0 & 2 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{array} \right)$

Solution:  $H : a + b = 2d$   
 $Cl : a = c$   
 $Na : b = c$   
 $O : b = d$       Each equation gets a row.

2. Suppose  $A$  is nilpotent, i.e.  $A^2 = 0$ . Which of the following is true?

- a.  $A + \mathbf{1}$  is invertible and  $(A + \mathbf{1})^{-1} = A - \mathbf{1}$
- b.  $A + \mathbf{1}$  is invertible and  $(A + \mathbf{1})^{-1} = \mathbf{1} - A$       Correct Answer
- c.  $A + \mathbf{1}$  is invertible and  $(A + \mathbf{1})^{-1} = \mathbf{1} - 2A$
- d.  $A + \mathbf{1}$  is invertible and  $(A + \mathbf{1})^{-1} = \mathbf{1} + A$
- e.  $A + \mathbf{1}$  is not invertible

Solution:  $(A + \mathbf{1})(\mathbf{1} - A) = A + \mathbf{1} - A^2 - A = \mathbf{1} - A^2 = \mathbf{1}$ .

3. Consider the vector space  $M(2,2)$  of  $2 \times 2$  matrices with the basis

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad E_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad E_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad E_4 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Which of the following are the components of the matrix  $A = \begin{pmatrix} 9 & 5 \\ 1 & 1 \end{pmatrix}$  relative to the  $E$  basis?

- a.  $(A)_E = \begin{pmatrix} 2 & 1 & 4 & 3 \end{pmatrix}^\top$
- b.  $(A)_E = \begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix}^\top$
- c.  $(A)_E = \begin{pmatrix} 4 & 5 & 2 & 3 \end{pmatrix}^\top$
- d.  $(A)_E = \begin{pmatrix} 5 & 4 & 3 & 2 \end{pmatrix}^\top$  Correct Answer
- e.  $(A)_E = \begin{pmatrix} 5 & -4 & 3 & -2 \end{pmatrix}^\top$

$$A = 2E_1 + 1E_2 + 4E_3 + 3E_4 = \begin{pmatrix} 3 & 7 \\ 1 & 1 \end{pmatrix} \quad A = 1E_1 + 2E_2 + 3E_3 + 4E_4 = \begin{pmatrix} 3 & 7 \\ -1 & -1 \end{pmatrix}$$

$$A = 4E_1 + 5E_2 + 2E_3 + 3E_4 = \begin{pmatrix} 9 & 5 \\ -1 & -1 \end{pmatrix} \quad A = 5E_1 + 4E_2 + 3E_3 + 2E_4 = \begin{pmatrix} 9 & 5 \\ 1 & 1 \end{pmatrix}$$

$$A = 5E_1 - 4E_2 + 3E_3 - 2E_4 = \begin{pmatrix} 1 & 1 \\ 5 & 9 \end{pmatrix}$$

4. Consider the vector space  $C_2([0,1])$  of real valued function on the interval  $[0,1]$  whose second derivatives exist and are continuous with the inner product  $\langle f, g \rangle = \int_{-1}^1 f(x)g(x)dx$ . Consider the subspace  $V = \text{Span}(x, x^2)$  spanned by the basis  $v_1 = x$ ,  $v_2 = x^2$ . Which of the following is an orthonormal basis for  $V$ ?

- a.  $u_1 = x$      $u_2 = x^2$
- b.  $u_1 = x$      $u_2 = x^2 - \frac{3}{4}x$
- c.  $u_1 = \sqrt{\frac{3}{2}}x$      $u_2 = \sqrt{\frac{5}{2}}x^2$  Correct Answer
- d.  $u_1 = \sqrt{\frac{5}{2}}x$      $u_2 = \sqrt{\frac{7}{2}}x^2$
- e.  $u_1 = \sqrt{2}x$      $u_2 = 4\sqrt{5}x^2 - 3\sqrt{5}x$

We apply Gram-Schmidt to  $v_1 = x$ ,  $v_2 = x^2$

$$w_1 = x \quad \langle w_1, w_1 \rangle = \langle x, x \rangle = \int_{-1}^1 x^2 dx = \frac{2}{3} \quad u_1 = \sqrt{\frac{3}{2}}x$$

$$\langle v_2, w_1 \rangle = \langle x^2, x \rangle = \int_{-1}^1 x^3 dx = 0 \quad w_2 = v_2 = x^2$$

$$\langle w_2, w_2 \rangle = \langle x^2, x^2 \rangle = \int_{-1}^1 x^4 dx = \frac{2}{5} \quad u_2 = \sqrt{\frac{5}{2}}x^2$$

5. Consider the second derivative linear operator  $L : P_6 \rightarrow P_6 : L(p) = \frac{d^2 p}{dx^2}$  on the space of polynomials of degree less than 6. Find the kernel,  $\text{Ker}(L)$ .

HINT: Let  $p = a + bx + cx^2 + dx^3 + ex^4 + fx^5$ .

- a.  $\text{Ker}(L) = \text{Span}(1)$
- b.  $\text{Ker}(L) = \text{Span}(1, x)$     **Correct Answer**
- c.  $\text{Ker}(L) = \text{Span}(1, x, x^2)$
- d.  $\text{Ker}(L) = \text{Span}(x^2, x^3, x^4, x^5)$
- e.  $\text{Ker}(L) = \text{Span}(x, x^2, x^3, x^4, x^5)$

$$\text{Ker}(L) = \{p \mid L(p) = 0\}$$

$$L(p) = 2c + 6dx + 12ex^2 + 20fx^3 = 0 \quad \Rightarrow \quad c = d = e = f = 0$$

$$\text{Ker}(L) = \{p = a + bx\} = \text{Span}(1, x)$$

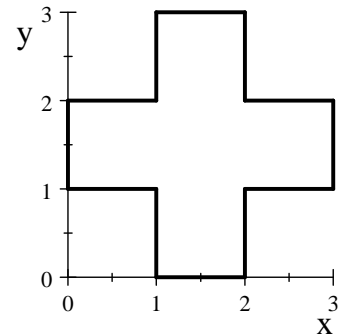
6. Consider the second derivative linear operator  $L : P_6 \rightarrow P_6 : L(p) = \frac{d^2 p}{dx^2}$  on the space of polynomials of degree less than 6. Find the image,  $\text{Im}(L)$ .

HINT: Let  $p = a + bx + cx^2 + dx^3 + ex^4 + fx^5$ .

- a.  $\text{Im}(L) = \text{Span}(1)$
- b.  $\text{Im}(L) = \text{Span}(1, x)$
- c.  $\text{Im}(L) = \text{Span}(1, x, x^2, x^3)$     **Correct Answer**
- d.  $\text{Im}(L) = \text{Span}(x^2, x^3, x^4, x^5)$
- e.  $\text{Im}(L) = \text{Span}(x, x^2, x^3, x^4, x^5)$

$$\text{Im}(L) = \{L(p)\} = \{2c + 6dx + 12ex^2 + 20fx^3\} = \text{Span}(1, x, x^2, x^3)$$

7. Compute the line integral  $\oint \vec{F} \cdot d\vec{s}$  clockwise around the complete boundary of the plus sign, shown at the right, for the vector field  $\vec{F} = (4x^3 + 2y, 4y^3 - 3x)$ .



- a. 25    **Correct Answer**
- b. 5
- c. 0
- d. -5
- e. -25

We let  $(P, Q) = \vec{F} = (4x^3 + 2y, 4y^3 - 3x)$  and apply Green's Theorem.

There is an extra minus sign because we need  $\oint \vec{F} \cdot d\vec{s}$  clockwise, but Green's Theorem expects counterclockwise:

$$\oint \vec{F} \cdot d\vec{s} = \oint P dx + Q dy = - \iint \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy = - \iint (-3 - 2) dx dy = 5 \cdot \text{Area} = 5 \cdot 5 = 25$$

8. Consider the vector space  $M(2,2)$  of  $2 \times 2$  matrices. Let  $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ . Consider the linear function,  $L : M(2,2) \rightarrow M(2,2) : L(X) = AX - XA$ . Which of the following is not an eigenvalue and corresponding eigenmatrix (eigenvector) of  $L$ ?

HINT: Let  $X = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$ .

Solution:

$$L(X) = L \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix} - \begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & -y \\ z & 0 \end{pmatrix}$$

a.  $\lambda = -1$   $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$$L \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} = -1 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

b.  $\lambda = 0$   $X = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

$$L \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

c.  $\lambda = 0$   $X = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$$L \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

d.  $\lambda = 1$   $X = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$$L \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = 1 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

e.  $\lambda = 2$   $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Correct Answer

$$L \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \neq 2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

9. (16 points) Which of the following is an inner product on  $\mathbb{R}^2$ ? If not, why not?

Put  $\times$ 's in the correct boxes. No part credit.

Let  $\vec{x} = (x_1, x_2)$ ,  $\vec{y} = (y_1, y_2)$ .

	$\langle \vec{x}, \vec{y} \rangle =$	Inner Product?		Why not?			
		Yes	No	Not Symmetric	Not Linear	Not Positive	Positive but Not Positive Definite
a.	$x_1y_1 + 2x_2y_2$	<input checked="" type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
b.	$x_1^2y_1^2 + 2x_2^2y_2^2$	<input type="checkbox"/>	<input checked="" type="checkbox"/>	<input type="checkbox"/>	<input checked="" type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
c.	$x_1y_1 + 2x_1y_2 + 2x_2y_2$	<input type="checkbox"/>	<input checked="" type="checkbox"/>	<input checked="" type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
d.	$x_1y_1 - x_1y_2 - x_2y_1 + x_2y_2$	<input type="checkbox"/>	<input checked="" type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input checked="" type="checkbox"/>
e.	$x_1y_1 + x_1y_2 + x_2y_1 + x_2y_2$	<input type="checkbox"/>	<input checked="" type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input checked="" type="checkbox"/>
f.	$x_1y_1 + x_1y_2 + x_2y_1 + 2x_2y_2$	<input checked="" type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
g.	$x_1y_1 - x_1y_2 - x_2y_1 + 2x_2y_2$	<input checked="" type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
h.	$x_1y_1 - x_2y_2$	<input type="checkbox"/>	<input checked="" type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input checked="" type="checkbox"/>	<input type="checkbox"/>

Solutions:

- a.  $\langle \vec{y}, \vec{x} \rangle = y_1x_1 + 2y_2x_2 = \langle \vec{x}, \vec{y} \rangle$      $\langle \vec{x}, a\vec{y} + b\vec{z} \rangle = a\langle \vec{x}, \vec{y} \rangle + b\langle \vec{x}, \vec{z} \rangle$   
 $\langle \vec{x}, \vec{x} \rangle = x_1^2 + 2x_2^2 \geq 0$  and  $= 0$  only if  $x_1 = x_2 = 0$
- b.  $\langle \vec{x}, a\vec{y} \rangle = a^2x_1^2y_1^2 + 2a^2x_2^2y_2^2 = a^2\langle \vec{x}, \vec{y} \rangle \neq a\langle \vec{x}, \vec{y} \rangle$
- c.  $\langle \vec{y}, \vec{x} \rangle = y_1x_1 + 2y_1x_2 + 2y_2x_2 \neq \langle \vec{x}, \vec{y} \rangle$
- d.  $\langle \vec{x}, \vec{x} \rangle = x_1^2 - 2x_1x_2 + x_2^2 = (x_1 - x_2)^2 \geq 0$  but can  $= 0$  if  $x_1 = x_2 \neq 0$
- e.  $\langle \vec{x}, \vec{x} \rangle = x_1^2 + 2x_1x_2 + x_2^2 = (x_1 + x_2)^2 \geq 0$  but can  $= 0$  if  $x_1 = -x_2 \neq 0$
- f.  $\langle \vec{y}, \vec{x} \rangle = y_1x_1 + y_1x_2 + y_2x_1 + 2y_2x_2 = \langle \vec{x}, \vec{y} \rangle$      $\langle \vec{x}, a\vec{y} + b\vec{z} \rangle = a\langle \vec{x}, \vec{y} \rangle + b\langle \vec{x}, \vec{z} \rangle$   
 $\langle \vec{x}, \vec{x} \rangle = x_1^2 + 2x_1x_2 + 2x_2^2 = (x_1 + x_2)^2 + x_2^2 \geq 0$  and  $= 0$  only if  $x_1 = x_2 = 0$
- g.  $\langle \vec{y}, \vec{x} \rangle = y_1x_1 - y_1x_2 - y_2x_1 + 2y_2x_2 = \langle \vec{x}, \vec{y} \rangle$      $\langle \vec{x}, a\vec{y} + b\vec{z} \rangle = a\langle \vec{x}, \vec{y} \rangle + b\langle \vec{x}, \vec{z} \rangle$   
 $\langle \vec{x}, \vec{x} \rangle = x_1^2 - 2x_1x_2 + 2x_2^2 = (x_1 - x_2)^2 + x_2^2 \geq 0$  and  $= 0$  only if  $x_1 = x_2 = 0$
- h.  $\langle \vec{x}, \vec{x} \rangle = x_1^2 - x_2^2 < 0$  if  $x_2 > x_1 > 0$

10. (20 points) Let  $M(2,3)$  be the vector space of  $2 \times 3$  matrices.

Consider the subspace  $V = \text{Span}(A_1, A_2, A_3, A_4)$  where

$$A_1 = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 3 & -2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 2 & 2 & 0 \\ 4 & 6 & -4 \end{pmatrix}, \quad A_3 = \begin{pmatrix} -1 & 0 & 2 \\ -3 & 0 & 0 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 1 & 3 & 4 \\ 0 & 9 & -6 \end{pmatrix}$$

Find a basis for  $V$ . What is the  $\dim V$ ?

Solution: To see if  $A_1, A_2, A_3, A_4$  are linearly independent, we write:

$$aA_1 + bA_2 + cA_3 + dA_4 = \mathbf{0} \text{ and solve for } a, b, c, d.$$

$$a \begin{pmatrix} 1 & 1 & 0 \\ 2 & 3 & -2 \end{pmatrix} + b \begin{pmatrix} 2 & 2 & 0 \\ 4 & 6 & -4 \end{pmatrix} + c \begin{pmatrix} -1 & 0 & 2 \\ -3 & 0 & 0 \end{pmatrix} + d \begin{pmatrix} 1 & 3 & 4 \\ 0 & 9 & -6 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} a + 2b - c + d & a + 2b + 3d & 2c + 4d \\ 2a + 4b - 3c & 3a + 6b + 9d & -2a - 4b - 6d \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{array}{l} \text{6 equations} \\ \text{4 unknowns} \end{array}$$

$$\left( \begin{array}{cccc|c} 1 & 2 & -1 & 1 & 0 \\ 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 2 & 4 & 0 \\ 2 & 4 & -3 & 0 & 0 \\ 3 & 6 & 0 & 9 & 0 \\ -2 & -4 & 0 & -6 & 0 \end{array} \right) \begin{array}{l} \\ R_2 - R_1 \\ \\ R_4 - 2R_1 \\ R_5 - 3R_1 \\ R_6 + 2R_1 \end{array} \Rightarrow \left( \begin{array}{cccc|c} 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 4 & 0 \\ 0 & 0 & -1 & -2 & 0 \\ 0 & 0 & 3 & 6 & 0 \\ 0 & 0 & -2 & -4 & 0 \end{array} \right) \begin{array}{l} R_1 + R_2 \\ \\ R_3 - 2R_2 \\ R_4 + R_2 \\ R_5 - 3R_2 \\ R_6 + 2R_2 \end{array} \Rightarrow \left( \begin{array}{cccc|c} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

$$\Rightarrow a + 2b + 3d = 0, \quad c + 2d = 0 \Rightarrow a = -2r - 3s \quad b = r \quad c = -2s \quad d = s$$

$(-2r - 3s)A_1 + rA_2 - 2sA_3 + sA_4 = \mathbf{0}$  So they are not linearly independent.

Further, with  $r = 1$  and  $s = 0$ :  $A_2 = 2A_1$  and with  $r = 0$  and  $s = 1$ :  $A_4 = 3A_1 + 2A_3$

So  $A_1, A_2$  also span and the above computation without columns 2 and 4 shows they are linearly independent. So  $A_1, A_2$  is a basis and  $\dim V = 2$ .

11. (28 points) Compute  $\iint_H \vec{F} \cdot d\vec{S}$  over the hemisphere  $z = \sqrt{25 - x^2 - y^2}$  oriented upward, for the vector field  $\vec{F} = (x^3 + 4y^2 + 4z^2, 4x^2 + y^3 + 4z^2, 4x^2 + 4y^2 + z^3)$ .  
 HINT: Use Gauss' Theorem by following these steps:

- Write out Gauss' Theorem for the Volume,  $V$ , which is the solid hemisphere  $0 \leq z \leq \sqrt{25 - x^2 - y^2}$ . Split up the boundary,  $\partial V$ , into two pieces, the hemisphere,  $H$ , and the disk,  $D$ , at the bottom. State orientations. Solve for the integral you want.
- Compute the volume integral using spherical coordinates.
- Compute the other surface integral over  $D$  by parametrizing the disk, computing the tangent vectors and normal vector, checking the orientation, evaluating the vector field on the disk and doing the integral.
- Solve for the original integral.

Solution:

a. Gauss' Theorem: 
$$\iiint_V \vec{\nabla} \cdot \vec{F} dV = \iint_{\partial V} \vec{F} \cdot d\vec{S} = \iint_H \vec{F} \cdot d\vec{S} + \iint_D \vec{F} \cdot d\vec{S}$$

$H$  is oriented upward.  $D$  is oriented downward.

$$\iint_H \vec{F} \cdot d\vec{S} = \iiint_V \vec{\nabla} \cdot \vec{F} dV - \iint_D \vec{F} \cdot d\vec{S}$$

- b. In spherical coordinates:  $dV = \rho^2 \sin \phi$  and  $\vec{\nabla} \cdot \vec{F} = 3x^2 + 3y^2 + 3z^2 = 3\rho^2$ .

$$\iiint_V \vec{\nabla} \cdot \vec{F} dV = \int_0^{2\pi} \int_0^{\pi/2} \int_0^5 3\rho^2 \rho^2 \sin \phi d\rho d\phi d\theta = 3 \cdot 2\pi \left[ -\cos \theta \right]_0^{\pi/2} \left[ \frac{\rho^5}{5} \right]_0^5 = 6\pi(1)5^4 = 3750\pi$$

- c. The disk may be parametrized by  $\vec{R}(r, \theta) = (r \cos \theta, r \sin \theta, 0)$ .

$$\begin{array}{l} \vec{e}_r = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos \theta & \sin \theta & 0 \\ -r \sin \theta & r \cos \theta & 0 \end{vmatrix} \\ \vec{e}_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos \theta & \sin \theta & 0 \\ -r \sin \theta & r \cos \theta & 0 \end{vmatrix} \end{array} \quad \begin{array}{l} \vec{N} = \vec{e}_r \times \vec{e}_\theta = \hat{i}(0) - \hat{j}(0) + \hat{k}(r \cos^2 \theta + r \sin^2 \theta) = (0, 0, r) \\ \text{Reverse to point down: } \vec{N} = (0, 0, -r) \end{array}$$

$$\vec{F} = (x^3 + 4y^2 + 4z^2, 4x^2 + y^3 + 4z^2, 4x^2 + 4y^2 + z^3) = (*, *, 4r^2 + 0^3) \quad \vec{F} \cdot \vec{N} = -4r^3$$

$$\iint_D \vec{F} \cdot d\vec{S} = \int_0^{2\pi} \int_0^5 \vec{F} \cdot \vec{N} dr d\theta = -\int_0^{2\pi} \int_0^5 4r^3 dr d\theta = -2\pi \left[ r^4 \right]_0^5 = -1250\pi$$

- d. 
$$\iint_H \vec{F} \cdot d\vec{S} = \iiint_V \vec{\nabla} \cdot \vec{F} dV - \iint_D \vec{F} \cdot d\vec{S} = 3750\pi - (-1250\pi) = 5000\pi$$