

Name _____

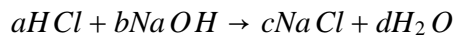
Math 311 Final Exam Version B Spring 2015
 Section 503 Solutions P. Yasskin

1-8	/40	10	/20
9	/16	11	/28
		Total	/104

Multiple Choice: 5 points each. No Part Credit

Work Out: Points indicated. Show all work.

1. Hydrochloric acid (HCl) and sodium hydroxide ($NaOH$) react to produce sodium chloride ($NaCl$) and water (H_2O) according to the chemical equation:



Which of the following is the augmented matrix which is used to solve this chemical equation?
 (Put the elements in the order H, Cl, Na, O .)

a. $\left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 & 0 \\ -2 & 0 & 0 & -1 & 0 \end{array} \right)$

b. $\left(\begin{array}{cccc|c} 1 & 1 & 0 & 2 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{array} \right)$

c. $\left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 2 & 0 & 0 & 1 & 0 \end{array} \right)$

d. $\left(\begin{array}{cccc|c} 1 & 1 & 0 & -2 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \end{array} \right)$

Correct Answer

Solution: $H : a + b = 2d$
 $Cl : a = c$
 $Na : b = c$
 $O : b = d$

Each equation gets a row.

2. Suppose A is nilpotent, i.e. $A^2 = 0$. Which of the following is true?

- a. $A + \mathbf{1}$ is invertible and $(A + \mathbf{1})^{-1} = A - \mathbf{1}$
- b. $A + \mathbf{1}$ is invertible and $(A + \mathbf{1})^{-1} = \mathbf{1} - 2A$
- c. $A + \mathbf{1}$ is invertible and $(A + \mathbf{1})^{-1} = \mathbf{1} - A$ Correct Answer
- d. $A + \mathbf{1}$ is invertible and $(A + \mathbf{1})^{-1} = \mathbf{1} + A$
- e. $A + \mathbf{1}$ is not invertible

Solution: $(A + \mathbf{1})(\mathbf{1} - A) = A + \mathbf{1} - A^2 - A = \mathbf{1} - A^2 = \mathbf{1}$.

3. Consider the vector space $M(2,2)$ of 2×2 matrices with the basis

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad E_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad E_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad E_4 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Which of the following are the components of the matrix $A = \begin{pmatrix} 9 & 5 \\ 1 & 1 \end{pmatrix}$ relative to the E basis?

- a. $(A)_E = \begin{pmatrix} 2 & 1 & 4 & 3 \end{pmatrix}^\top$
- b. $(A)_E = \begin{pmatrix} 5 & 4 & 3 & 2 \end{pmatrix}^\top$ **Correct Answer**
- c. $(A)_E = \begin{pmatrix} 4 & 5 & 2 & 3 \end{pmatrix}^\top$
- d. $(A)_E = \begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix}^\top$
- e. $(A)_E = \begin{pmatrix} 5 & -4 & 3 & -2 \end{pmatrix}^\top$

$$A = 2E_1 + 1E_2 + 4E_3 + 3E_4 = \begin{pmatrix} 3 & 7 \\ 1 & 1 \end{pmatrix}$$

$$A = 5E_1 + 4E_2 + 3E_3 + 2E_4 = \begin{pmatrix} 9 & 5 \\ 1 & 1 \end{pmatrix}$$

$$A = 4E_1 + 5E_2 + 2E_3 + 3E_4 = \begin{pmatrix} 9 & 5 \\ -1 & -1 \end{pmatrix}$$

$$A = 1E_1 + 2E_2 + 3E_3 + 4E_4 = \begin{pmatrix} 3 & 7 \\ -1 & -1 \end{pmatrix}$$

$$A = 5E_1 - 4E_2 + 3E_3 - 2E_4 = \begin{pmatrix} 1 & 1 \\ 5 & 9 \end{pmatrix}$$

4. Consider the vector space $C_2([0,1])$ of real valued function on the interval $[0,1]$ whose second derivatives exist and are continuous with the inner product $\langle f, g \rangle = \int_{-1}^1 f(x)g(x)dx$. Consider the subspace $V = \text{Span}(x, x^2)$ spanned by the basis $v_1 = x$, $v_2 = x^2$. Which of the following is an orthonormal basis for V ?

- a. $u_1 = x$ $u_2 = x^2$
- b. $u_1 = \sqrt{\frac{5}{2}}x$ $u_2 = \sqrt{\frac{7}{2}}x^2$
- c. $u_1 = x$ $u_2 = x^2 - \frac{3}{4}x$
- d. $u_1 = \sqrt{\frac{3}{2}}x$ $u_2 = \sqrt{\frac{5}{2}}x^2$ **Correct Answer**
- e. $u_1 = \sqrt{2}x$ $u_2 = 4\sqrt{5}x^2 - 3\sqrt{5}x$

We apply Gram-Schmidt to $v_1 = x$, $v_2 = x^2$

$$w_1 = x \quad \langle w_1, w_1 \rangle = \langle x, x \rangle = \int_{-1}^1 x^2 dx = \frac{2}{3} \quad u_1 = \sqrt{\frac{3}{2}}x$$

$$\langle v_2, w_1 \rangle = \langle x^2, x \rangle = \int_{-1}^1 x^3 dx = 0 \quad w_2 = v_2 = x^2$$

$$\langle w_2, w_2 \rangle = \langle x^2, x^2 \rangle = \int_{-1}^1 x^4 dx = \frac{2}{5} \quad u_2 = \sqrt{\frac{5}{2}}x^2$$

5. Consider the second derivative linear operator $L : P_6 \rightarrow P_6 : L(p) = \frac{d^2p}{dx^2}$ on the space of polynomials of degree less than 6. Find the image, $\text{Im}(L)$.

HINT: Let $p = a + bx + cx^2 + dx^3 + ex^4 + fx^5$.

- a. $\text{Im}(L) = \text{Span}(1)$
- b. $\text{Im}(L) = \text{Span}(1, x)$
- c. $\text{Im}(L) = \text{Span}(1, x, x^2, x^3)$ **Correct Answer**
- d. $\text{Im}(L) = \text{Span}(x^2, x^3, x^4, x^5)$
- e. $\text{Im}(L) = \text{Span}(x, x^2, x^3, x^4, x^5)$

$$\text{Im}(L) = \{L(p)\} = \{2c + 6dx + 12ex^2 + 20fx^3\} = \text{Span}(1, x, x^2, x^3)$$

6. Consider the second derivative linear operator $L : P_6 \rightarrow P_6 : L(p) = \frac{d^2p}{dx^2}$ on the space of polynomials of degree less than 6. Find the kernel, $\text{Ker}(L)$.

HINT: Let $p = a + bx + cx^2 + dx^3 + ex^4 + fx^5$.

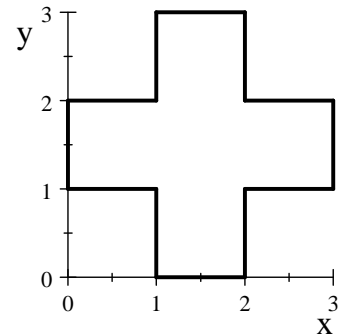
- a. $\text{Ker}(L) = \text{Span}(1)$
- b. $\text{Ker}(L) = \text{Span}(1, x)$ **Correct Answer**
- c. $\text{Ker}(L) = \text{Span}(1, x, x^2)$
- d. $\text{Ker}(L) = \text{Span}(x^2, x^3, x^4, x^5)$
- e. $\text{Ker}(L) = \text{Span}(x, x^2, x^3, x^4, x^5)$

$$\text{Ker}(L) = \{p \mid L(p) = 0\}$$

$$L(p) = 2c + 6dx + 12ex^2 + 20fx^3 = 0 \quad \Rightarrow \quad c = d = e = f = 0$$

$$\text{Ker}(L) = \{p = a + bx\} = \text{Span}(1, x)$$

7. Compute the line integral $\oint \vec{F} \cdot d\vec{s}$ clockwise around the complete boundary of the plus sign, shown at the right, for the vector field $\vec{F} = (4x^3 + 2y, 4y^3 - 3x)$.



- a. -25
- b. -5
- c. 0
- d. 5
- e. 25 **Correct Answer**

We let $(P, Q) = \vec{F} = (4x^3 + 2y, 4y^3 - 3x)$ and apply Green's Theorem.

There is an extra minus sign because we need $\oint \vec{F} \cdot d\vec{s}$ clockwise, but Green's Theorem expects counterclockwise:

$$\oint \vec{F} \cdot d\vec{s} = \oint P dx + Q dy = - \iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy = - \iint (-3 - 2) dx dy = 5 \cdot \text{Area} = 5 \cdot 5 = 25$$

8. Consider the vector space $M(2,2)$ of 2×2 matrices. Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. Consider the linear function, $L : M(2,2) \rightarrow M(2,2) : L(X) = AX - XA$. Which of the following is not an eigenvalue and corresponding eigenmatrix (eigenvector) of L ?

HINT: Let $X = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$.

Solution:

$$L(X) = L \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix} - \begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & -y \\ z & 0 \end{pmatrix}$$

a. $\lambda = 2$	$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	Correct Answer	$L \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \neq 2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
b. $\lambda = 1$	$X = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$		$L \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = 1 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$
c. $\lambda = 0$	$X = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$		$L \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$
d. $\lambda = 0$	$X = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$		$L \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$
e. $\lambda = -1$	$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$		$L \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} = -1 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

9. (16 points) Which of the following is an inner product on \mathbb{R}^2 ? If not, why not?

Put \times 's in the correct boxes. No part credit.

Let $\vec{x} = (x_1, x_2)$, $\vec{y} = (y_1, y_2)$.

		Why not?					
		Inner Product?		Not Symmetric	Not Linear	Not Positive	Positive but Not Positive Definite
$\langle \vec{x}, \vec{y} \rangle =$		Yes	No				
a.	$x_1y_1 + 2x_2y_2$	\times					
b.	$x_1y_1 + 2x_1y_2 + 2x_2y_2$		\times	\times			
c.	$x_1^2y_1^2 + 2x_2^2y_2^2$		\times		\times		
d.	$x_1y_1 + x_1y_2 + x_2y_1 + 2x_2y_2$	\times					
e.	$x_1y_1 - x_1y_2 - x_2y_1 + 2x_2y_2$	\times					
f.	$x_1y_1 - x_1y_2 - x_2y_1 + x_2y_2$		\times				\times
g.	$x_1y_1 + x_1y_2 + x_2y_1 + x_2y_2$		\times				\times
h.	$x_1y_1 - x_2y_2$		\times			\times	

Solutions:

- a. $\langle \vec{y}, \vec{x} \rangle = y_1x_1 + 2y_2x_2 = \langle \vec{x}, \vec{y} \rangle$ $\langle \vec{x}, a\vec{y} + b\vec{z} \rangle = a\langle \vec{x}, \vec{y} \rangle + b\langle \vec{x}, \vec{z} \rangle$
 $\langle \vec{x}, \vec{x} \rangle = x_1^2 + 2x_2^2 \geq 0$ and $= 0$ only if $x_1 = x_2 = 0$
- b. $\langle \vec{y}, \vec{x} \rangle = y_1x_1 + 2y_1x_2 + 2y_2x_2 \neq \langle \vec{x}, \vec{y} \rangle$
- c. $\langle \vec{x}, a\vec{y} \rangle = a^2x_1^2y_1^2 + 2a^2x_2^2y_2^2 = a^2\langle \vec{x}, \vec{y} \rangle \neq a\langle \vec{x}, \vec{y} \rangle$
- d. $\langle \vec{y}, \vec{x} \rangle = y_1x_1 + y_1x_2 + y_2x_1 + 2y_2x_2 = \langle \vec{x}, \vec{y} \rangle$ $\langle \vec{x}, a\vec{y} + b\vec{z} \rangle = a\langle \vec{x}, \vec{y} \rangle + b\langle \vec{x}, \vec{z} \rangle$
 $\langle \vec{x}, \vec{x} \rangle = x_1^2 + 2x_1x_2 + 2x_2^2 = (x_1 + x_2)^2 + x_2^2 \geq 0$ and $= 0$ only if $x_1 = x_2 = 0$
- e. $\langle \vec{y}, \vec{x} \rangle = y_1x_1 - y_1x_2 - y_2x_1 + 2y_2x_2 = \langle \vec{x}, \vec{y} \rangle$ $\langle \vec{x}, a\vec{y} + b\vec{z} \rangle = a\langle \vec{x}, \vec{y} \rangle + b\langle \vec{x}, \vec{z} \rangle$
 $\langle \vec{x}, \vec{x} \rangle = x_1^2 - 2x_1x_2 + 2x_2^2 = (x_1 - x_2)^2 + x_2^2 \geq 0$ and $= 0$ only if $x_1 = x_2 = 0$
- f. $\langle \vec{x}, \vec{x} \rangle = x_1^2 - 2x_1x_2 + x_2^2 = (x_1 - x_2)^2 \geq 0$ but can $= 0$ if $x_1 = x_2 \neq 0$
- g. $\langle \vec{x}, \vec{x} \rangle = x_1^2 + 2x_1x_2 + x_2^2 = (x_1 + x_2)^2 \geq 0$ but can $= 0$ if $x_1 = -x_2 \neq 0$
- h. $\langle \vec{x}, \vec{x} \rangle = x_1^2 - x_2^2 < 0$ if $x_2 > x_1 > 0$

10. (20 points) Let $M(2,3)$ be the vector space of 2×3 matrices.

Consider the subspace $V = \text{Span}(A_1, A_2, A_3, A_4)$ where

$$A_1 = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 3 & -2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 2 & 2 & 0 \\ 4 & 6 & -4 \end{pmatrix}, \quad A_3 = \begin{pmatrix} -1 & 0 & 2 \\ -3 & 0 & 0 \end{pmatrix}, \quad A_4 = \begin{pmatrix} -1 & 2 & 6 \\ -5 & 6 & -4 \end{pmatrix}$$

Find a basis for V . What is the $\dim V$?

Solution: To see if A_1, A_2, A_3, A_4 are linearly independent, we write:

$$aA_1 + bA_2 + cA_3 + dA_4 = \mathbf{0} \text{ and solve for } a, b, c, d.$$

$$a \begin{pmatrix} 1 & 1 & 0 \\ 2 & 3 & -2 \end{pmatrix} + b \begin{pmatrix} 2 & 2 & 0 \\ 4 & 6 & -4 \end{pmatrix} + c \begin{pmatrix} -1 & 0 & 2 \\ -3 & 0 & 0 \end{pmatrix} + d \begin{pmatrix} -1 & 2 & 6 \\ -5 & 6 & -4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} a + 2b - c - d & a + 2b + 2d & 2c + 6d \\ 2a + 4b - 3c - 5d & 3a + 6b + 6d & -2a - 4b - 4d \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{array}{l} \text{6 equations} \\ \text{4 unknowns} \end{array}$$

$$\left(\begin{array}{cccc|c} 1 & 2 & -1 & -1 & 0 \\ 1 & 2 & 0 & 2 & 0 \\ 0 & 0 & 2 & 6 & 0 \\ 2 & 4 & -3 & -5 & 0 \\ 3 & 6 & 0 & 6 & 0 \\ -2 & -4 & 0 & -4 & 0 \end{array} \right) \begin{array}{l} \\ R_2 - R_1 \\ \\ R_4 - 2R_1 \\ R_5 - 3R_1 \\ R_6 + 2R_1 \end{array} \Rightarrow \left(\begin{array}{cccc|c} 1 & 2 & -1 & -1 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 2 & 6 & 0 \\ 0 & 0 & -1 & -3 & 0 \\ 0 & 0 & 3 & 9 & 0 \\ 0 & 0 & -2 & -6 & 0 \end{array} \right) \begin{array}{l} R_1 + R_2 \\ \\ R_3 - 2R_2 \\ R_4 + R_2 \\ R_5 - 3R_2 \\ R_6 + 2R_2 \end{array} \Rightarrow \left(\begin{array}{cccc|c} 1 & 2 & 0 & 2 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

$$\Rightarrow a + 2b + 2d = 0, \quad c + 3d = 0 \Rightarrow a = -2r - 2s \quad b = r \quad c = -3s \quad d = s$$

$(-2r - 2s)A_1 + rA_2 - 3sA_3 + sA_4 = \mathbf{0}$ So they are not linearly independent.

Further, with $r = 1$ and $s = 0$: $A_2 = 2A_1$ and with $r = 0$ and $s = 1$: $A_4 = 2A_1 + 3A_3$

So A_1, A_2 also span and the above computation without columns 2 and 4 shows they are linearly independent. So A_1, A_2 is a basis and $\dim V = 2$.

11. (28 points) Compute $\iint_H \vec{F} \cdot d\vec{S}$ over the hemisphere $z = \sqrt{25 - x^2 - y^2}$ oriented upward, for the vector field $\vec{F} = (x^3 - 4y^2 - 4z^2, -4x^2 + y^3 - 4z^2, -4x^2 - 4y^2 + z^3)$.
 HINT: Use Gauss' Theorem by following these steps:

- Write out Gauss' Theorem for the Volume, V , which is the solid hemisphere $0 \leq z \leq \sqrt{25 - x^2 - y^2}$. Split up the boundary, ∂V , into two pieces, the hemisphere, H , and the disk, D , at the bottom. State orientations. Solve for the integral you want.
- Compute the volume integral using spherical coordinates.
- Compute the other surface integral over D by parametrizing the disk, computing the tangent vectors and normal vector, checking the orientation, evaluating the vector field on the disk and doing the integral.
- Solve for the original integral.

Solution:

a. Gauss' Theorem: $\iiint_V \vec{\nabla} \cdot \vec{F} dV = \iint_{\partial V} \vec{F} \cdot d\vec{S} = \iint_H \vec{F} \cdot d\vec{S} + \iint_D \vec{F} \cdot d\vec{S}$

H is oriented upward. D is oriented downward.

$$\iint_H \vec{F} \cdot d\vec{S} = \iiint_V \vec{\nabla} \cdot \vec{F} dV - \iint_D \vec{F} \cdot d\vec{S}$$

- b. In spherical coordinates: $dV = \rho^2 \sin \phi$ and $\vec{\nabla} \cdot \vec{F} = 3x^2 + 3y^2 + 3z^2 = 3\rho^2$.

$$\iiint_V \vec{\nabla} \cdot \vec{F} dV = \int_0^{2\pi} \int_0^{\pi/2} \int_0^5 3\rho^2 \rho^2 \sin \phi d\rho d\phi d\theta = 3 \cdot 2\pi \left[-\cos \theta \right]_0^{\pi/2} \left[\frac{\rho^5}{5} \right]_0^5 = 6\pi(1)5^4 = 3750\pi$$

- c. The disk may be parametrized by $\vec{R}(r, \theta) = (r \cos \theta, r \sin \theta, 0)$.

$$\begin{array}{l} \vec{e}_r = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos \theta & \sin \theta & 0 \\ -r \sin \theta & r \cos \theta & 0 \end{vmatrix} \\ \vec{e}_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos \theta & \sin \theta & 0 \\ -r \sin \theta & r \cos \theta & 0 \end{vmatrix} \end{array} \quad \begin{array}{l} \vec{N} = \vec{e}_r \times \vec{e}_\theta = \hat{i}(0) - \hat{j}(0) + \hat{k}(r \cos^2 \theta + r \sin^2 \theta) = (0, 0, r) \\ \text{Reverse to point down: } \vec{N} = (0, 0, -r) \end{array}$$

$$\vec{F} = (x^3 - 4y^2 - 4z^2, -4x^2 + y^3 - 4z^2, -4x^2 - 4y^2 + z^3) = (*, *, -4r^2 + 0^3) \quad \vec{F} \cdot \vec{N} = 4r^3$$

$$\iint_D \vec{F} \cdot d\vec{S} = \int_0^{2\pi} \int_0^5 \vec{F} \cdot \vec{N} dr d\theta = \int_0^{2\pi} \int_0^5 4r^3 dr d\theta = 2\pi \left[r^4 \right]_0^5 = 1250\pi$$

- d. $\iint_H \vec{F} \cdot d\vec{S} = \iiint_V \vec{\nabla} \cdot \vec{F} dV - \iint_D \vec{F} \cdot d\vec{S} = 3750\pi - 1250\pi = 2500\pi$